

MA201: Complex Analysis

Assignment 5

(Morera's theorem, Power series, and identity theorem)

July - November, 2024

1. Suppose f is analytic on the open unit disc D and it satisfies $|f(z)| \leq 1$ for all $z \in D$. Show that $|f'(0)| \leq 1$.
2. If $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous and analytic on $\mathbb{C} \setminus [-1, 1]$, then show that f is entire.
3. Define $F(z) = \int_0^1 \sin t^2 e^{-itz} dt$. Show that F is entire and satisfying $|F(z)| \leq A e^{B|y|}$ for $z = x + iy$ and for some positive constants A and B .
4. Find all the entire functions f such that $f(x) = e^x$ for all x in \mathbb{R} .
5. Let f and g be analytic functions on a domain D in \mathbb{C} . If $\bar{f}g$ is analytic, then show that either f is constant or $g \equiv 0$.
6. Let f be an entire function such that $\lim_{z \rightarrow \infty} \left| \frac{f(z)}{z} \right| = 0$. Show that f is constant.
7. Find the radius of convergence of the following power series:
 - (a) $\sum_{n \geq 0} z^{n!}$
 - (b) $\sum_{n \geq 0} 2^{n^2} z^n$
 - (c) $\sum_{n \geq 0} \frac{(-1)^n}{n} z^{n(n+1)}$
 - (d) $\sum_{n \geq 0} a_n z^n$ where $a_n = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 3^n & \text{if } n \text{ is even.} \end{cases}$
8. Find the power series expansion of the function $f(z) = \cos^2 z$ about 0.