MA201: Complex Analysis

Assignment 5

(Morera's theorem, Power series, and identity theorem)

July - November, 2024

1. Suppose f is analytic on the open unit disc D and it satisfies $|f(z)| \leq 1$ for all $z \in D$. Show that $|f'(0)| \leq 1$.

   Answer: (Hint: Use Cauchy's integral formula)

2. If $f : \mathbb{C} \to \mathbb{C}$ is continuous and analytic on $\mathbb{C} \setminus [-1, 1]$, then show that f is entire.

   Answer: (Hint: Use Morera's theorem)

3. Define $F(z) = \int_0^1 \sin t^2 \, e^{-itz} dt$. Show that F is entire and satisfying $|F(z)| \leq A\, e^{B|y|}$ for $z = x + iy$ and for some positive constants A and B.

   Answer: (Hint: Use Morera's theorem)

4. Find all the entire functions f such that $f(x) = e^x$ for all x in $\mathbb{R}$.

   Answer: (Hint: Use identity theorem)

5. Let f and g be analytic functions on a domain D in $\mathbb{C}$. If $\bar{f}\, g$ is analytic, then show that either f is constant or $g \equiv 0$.

   Answer: Suppose $g \not\equiv 0$. Then there exists $z_0 \in D$ and $\delta > 0$ such that $g \neq 0$ on $B(z_0, \delta)$. Since $\bar{f}\, g$ is analytic, it follows that $(\bar{f} g)\frac{1}{g} = \bar{f}$ is analytic on $B(z_0, \delta)$. So f and $\bar{f}$ both are analytic $B(z_0, \delta)$. By Cauchy-Riemann equations, f is constant on $B(z_0, \delta)$. Hence, by identity theorem, f is constant on D.

6. Let f be an entire function such that $\lim_{z\to\infty} \left|\frac{f(z)}{z}\right| = 0$. Show that f is constant.

   Answer: Given that $\lim_{z\to\infty} \left|\frac{f(z)}{z}\right| = 0$. For every $\epsilon > 0$, there exists a $M > 0$ such that $\left|\frac{f(z)}{z}\right| < \epsilon$, whenever $|z| > M$. That is,

   $$|f(z)| < \epsilon|z|, \text{ whenever } |z| > M \Longrightarrow f(z) = az + b$$

   for some constants $a, b \in \mathbb{C}$. Once again

   $$\lim_{z\to\infty} \left|\frac{f(z)}{z}\right| = \lim_{z\to\infty} \left|\frac{az+b}{z}\right| = \lim_{z\to\infty} \left|a + \frac{b}{z}\right| = 0.$$

   So $a = 0$ and hence f is constant.

7. Find the radius of convergence of the following power series:

   (a) $\sum_{n\geq 0} z^{n!}$ (**R=1**)

   (b) $\sum_{n\geq 0} 2^{n^2} z^n$ (**R=0**)

   (c) $\sum_{n\geq 0} \frac{(-1)^n}{n} z^{n(n+1)}$ (**R=1**)

(d) $\sum_{n\geq 0} a_n z^n$ where $a_n = \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 3^n & \text{if } n \text{ is even.} \end{cases}$ $(\mathbf{R} = \mathbf{\frac{1}{3}})$

8. Find the power series expansion of the function $f(z) = \cos^2 z$ about 0.

Answer: We know that $f(z) = \cos^2 z = (1 + \cos 2z)/2$ and

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!}$$

for $z \in \mathbb{C}$. Therefore,

$$f(z) = \cos^2 z = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(2z)^{2n}}{2n!} \qquad \text{for } z \in \mathbb{C}\,.$$