MA201: Complex Analysis

Assignment 6

(Taylor and Laurent expansions, identity theorem, and maximum modulus theorem)

July - November, 2024

1. Is there a polynomial $P(z)$ such that $P(z)e^{\frac{1}{z}}$ is an entire function? Justify your answer.
2. Find the Laurent series of the function $f(z) = \exp\left(z + \frac{1}{z}\right)$ around 0. Further, show that for all $n \geq 0$
$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta d\theta = \sum_{k=0}^{\infty} \frac{1}{(n+k)!k!}.$$
3. If f and g are entire functions such that $g\bar{f}$ is entire, then either f is constant or $g \equiv 0$.
4. Find the Laurent series expansion of the following functions about the given points $z = z_0$ or in the given region (specify the region in which the expansion is valid wherever is necessary).
   (a) $z^2 \exp(1/z)$ in the neighborhood of $z = 0$
   (b) $\dfrac{1}{z^2+1}$ in the neighborhood of $z = -i$
   (c) $f(z) = \dfrac{z+3}{z(z^2 - z - 2)}$ for $0 < |z| < 1$ and for $1 < |z| < 2$.
5. Use the maximum modulus theorem to prove the fundamental theorem of algebra.
6. Let f be a bounded analytic function on the right half plane (RHP). If f is continuous on the imaginary axis and satisfies $\sup_{y\in\mathbb{R}} |f(iy)| \leq M$, then show that $|f(z)| \leq M$ on the RHP. (Hint: Use maximum modulus theorem to $g_\epsilon(z) = (z+1)^{-\epsilon} f(z)$ on an appropriate semi-disc.)