MA201: Complex Analysis

Assignment 6: Hint/ model solutions

(Taylor and Laurent expansions, identity theorem, and maximum modulus theorem)

July - November 2024

1. Is there a polynomial $P(z)$ such that $P(z)e^{\frac{1}{z}}$ is an entire function? Justify your answer.

   Answer: No. Notice that there are infinitely many terms in the principal part of $e^{\frac{1}{z}}$ in the Laurent series expansion about 0.

   If we multiply it with any polynomial $P(z)$. then also $P(z)e^{\frac{1}{z}}$ has infinitely many terms in the principal part of the Laurent series about 0.

2. Find the Laurent series of the function $f(z) = \exp\left(z + \frac{1}{z}\right)$ around 0. Further, show that for all $n \geq 0$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta d\theta = \sum_{k=0}^{\infty}\frac{1}{(n+k)!k!}.$$

   Answer: Laurent Series of $f(z) = \exp\left(z + \frac{1}{z}\right)$ around 0 is given by

$$\begin{aligned}\exp\left(z+\frac{1}{z}\right) &= \left(\sum_{m=0}^{\infty}\frac{z^m}{m!}\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!z^k}\right)\\ &= \sum_{n=-\infty}^{\infty}\left(\sum_{m-k=n(k,m\geq 0)}\frac{1}{k!m!}\right)z^n\\ &= \sum_{n=-\infty}^{\infty}\left(\sum_{k\geq\max(0,-n)}\frac{1}{(n+k)!k!}\right)z^n.\end{aligned}$$

   It follows from the above expression and the uniqueness of the Laurent series that if $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ then

$$a_n = \sum_{k\geq\max(0,-n)}\frac{1}{(n+k)!k!} = \frac{1}{2\pi i}\int_C \frac{f(w)}{w^{n+1}}dw$$

   where C is the unit circle. Since the imaginary part of a_n's is 0 thus,

$$a_n = \Re\left(\frac{1}{2\pi i}\int_C\frac{f(w)}{w^{n+1}}dw\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta d\theta.$$

   For $n \geq 0$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta d\theta = \sum_{k=0}^{\infty}\frac{1}{(n+k)!k!}.$$

3. If f and g are entire functions such that $g\bar{f}$ is entire, then either f is constant or $g \equiv 0$.

   Answer: Let $g \not\equiv 0$. That means there is a point $a \in \mathbb{C}$ such that $g(a) \neq 0$. So there is a $r > 0$ such that $g(z) \neq 0$ for every $z \in B(a, r)$ (as g is continuous at a). Then $\bar{f}(z) = \bar{f}(z)g(z)\frac{1}{g(z)}$ is analytic on $B(a, r)$. But f is analytic on $B(a, r)$. Then f must be a constant on $B(a, r)$. By the identity theorem f is constant on $\mathbb{C}$.

4. Find the Laurent series expansion of the following functions about the given points $z = z_0$ or in the given region (specify the region where the expansion is valid wherever necessary).

(a) $z^2 \exp(1/z)$ in the neighborhood of $z = 0$

(b) $\dfrac{1}{z^2+1}$ in the neighborhood of $z = -i$

(c) $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ for $0 < |z| < 1$ and for $1 < |z| < 2$.

Answer: (a) $f(z) = z^2 \exp(1/z)$

About $z = 0$:

$$\begin{aligned} z^2 e^{\frac{1}{z}} &= z^2 \times \left(1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \cdots\right) \\ &= z^2 + \frac{1}{1!}z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots \qquad \text{for } |z| > 0\,. \end{aligned}$$

(b) $\dfrac{1}{z^2+1}$

About $z = -i$:

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{(z-(-i))(z+i-2i)} = \frac{1}{(z-(-i))(-2i)\left(1-\left(\frac{z+i}{2i}\right)\right)}$$

$$\begin{aligned} \frac{1}{z^2+1} &= \frac{1}{(z+i)(-2i)}\left[1-\left(\frac{z+i}{2i}\right)\right]^{-1} \\ &= \frac{1}{(z+i)(-2i)}\left[\sum_{n=0}^{\infty}\left(\frac{z+i}{2i}\right)^n\right] \\ &= (-1)\left[\frac{1}{(z+i)(2i)} + \sum_{n=1}^{\infty}\frac{(z+i)^{n-1}}{(2i)^{n+1}}\right] \end{aligned}$$

The above series converges for $0 < |z+i| < 2$.

(c) $f(z) = \dfrac{z+3}{z(z^2-z-2)}$

In the domain $0 < |z| < 1$:

$$f(z) = \frac{z+3}{z(z^2-z-2)} = \frac{-3}{2z} + \frac{2}{3(z+1)} + \frac{5}{6(z-2)}\,.$$

$$\begin{aligned} \frac{z+3}{z(z^2-z-2)} &= \frac{-3}{2z} + \frac{2}{3}[1+z]^{-1} - \frac{5}{12}\left[1-\left(\frac{z}{2}\right)\right]^{-1} \\ &= \frac{-3}{2z} + \frac{2}{3}\left[\sum_{n=0}^{\infty}(-1)^n z^n\right] - \frac{5}{12}\left[\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right] \\ &= \frac{-3}{2z} + \sum_{n=0}^{\infty}\left(\frac{(-1)^n 2}{3} + \frac{(-5)}{(12)(2)^n}\right) z^n \end{aligned}$$

The above series converges in the domain $0 < |z| < 1$.

In the domain $1 < |z| < 2$:

$$\begin{aligned}\frac{z+3}{z(z^2-z-2)} &= \frac{-3}{2z} + \frac{2}{3z}\left[1+\left(\frac{1}{z}\right)\right]^{-1} - \frac{5}{12}\left[1-\left(\frac{z}{2}\right)\right]^{-1} \\ &= \frac{-3}{2z} + \frac{2}{3z}\left[\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n\right] - \frac{5}{12}\left[\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right]\end{aligned}$$

The above series converges in the domain $1 < |z| < 2$.

5. Use the maximum modulus theorem (MMT) to prove the fundamental theorem of algebra.

   Answer: If $P(z) \neq 0$ for all $z \in \mathbb{C}$, then $f(z) = \frac{1}{p(z)}$ is entire and by the maximum modulus theorem, $\sup_{|z|\leq R} |f(z)| = |f(z_0)| = M$, where $|z_0| = R$. On the other hand, $|f(z)| \to 0$ when $z \to \infty$. Hence, for $0 < \epsilon < M$, we can choose $R_1 > R$ such that $|f(z)| < \epsilon$ whenever $|z| \geq R_1$. Again, applying the MMT on $\overline{B(0, R_1)}$, we get $\sup_{|z|\leq R_1} |f(z)| = |f(z_1)| \leq \epsilon < M$, where $|z_1| = R_1$. This implies that the maximum modulus is attained the interior point z_0 of $B(0, R_1)$. By MMT, it turns out that f is constant.

6. Let f be a bounded analytic function on the (open) right half-plane (RHP). If f is continuous on the imaginary axis and satisfies $\sup_{y\in\mathbb{R}} |f(iy)| \leq M$, then show that $|f(z)| \leq M$ on the RHP. (Hint: Use maximum modulus theorem to $g_\epsilon(z) = (z+1)^{-\epsilon} f(z)$ on an appropriate semi-disc.)

   Answer: Let $|f(z)| \leq K$ for $z \in$ RHP. For $\epsilon > 0$, write $g_\epsilon(z) = (z+1)^{-\epsilon} f(z)$. On the half circle $S(0, R) = \{z \in \text{RHP} : |z| = R\}$, we get $|g_\epsilon(z)| \leq (R-1)^{-\epsilon} K \to 0$, when $R \to \infty$. On the other hand, when $z \in [-iR, iR]$, we have $|g_\epsilon(z)| \leq (R-1)^{-\epsilon} M$, which is holds true for each $\epsilon > 0$. Hence $|f(z)| \leq M$ for all $z \in [-iR, iR] \cup S(0, R)$. By the maximum modulus theorem, $|f(z)| \leq M$ on the region $[-iR, iR] \cup \{z \in \text{RHP} : |z| \leq R\}$ for sufficiently large R. Letting $R \to \infty$, we get $|f(z)| \leq M$ for all z in the RHP.

   Remark: One can apply MMT in the case when $|f(z)| \leq |f(z_0)|$ for some z_0 in the domain. Only then we can think of the location of z_0 either in the domain or on the boundary of the domain.

   Let's focus on the right half-plane (RHP). It's evident that $f(z) = \dfrac{z}{1+z}$ meets all the conditions of Q6. Clearly $|f(z)| < 1$ on the RHP. However, the value 1 is not attended by $|f(z)|$ at any point on the RHP. Thus, it is necessary to utilize a technique like the one mentioned above.