MA201: Complex Analysis

Assignment 7

(Residue theorem, argument principle, Rouche's theorem and Mӧbius transforms)

July - November 2024

1. Evaluate $\displaystyle\int_{|z-\frac{\pi}{2}|=\frac{\pi}{2}} \frac{z}{\cos z}\, dz.$

   Answer: Since $\cos z$ has simple zeros at $\dfrac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$, the function $f(z) = \dfrac{z}{\cos z}$ has simple poles at $\dfrac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$. In the interior enclosed by the contour $|z - \frac{\pi}{2}| = \frac{\pi}{2}$, the function f has only one pole $z = \frac{\pi}{2}$, which is a simple pole. Hence,

$$\text{Res}\left(f, \frac{\pi}{2}\right) = \left[\frac{z}{-\sin z}\right]_{z=\frac{\pi}{2}} = -\frac{\pi}{2}\ .$$

   By the Cauchy's residue theorem, we get

$$\int_{|z-\frac{\pi}{2}|} \frac{z\, dz}{\cos z} = 2\pi i \,\text{Res}\left(f, \frac{\pi}{2}\right) = 2\pi i \times (-\frac{\pi}{2}) = -\pi^2\, i\ .$$

2. Using the Cauchy's residue theorem, evaluate $\displaystyle\int_{|z|=2} \frac{(z^2+3z+2)}{(z^3-z^2)}\, dz.$

   Answer: Let $f(z) = \dfrac{(z^2+3z+2)}{(z^3-z^2)} = \dfrac{(z^2+3z+2)}{z^2(z-1)}$. Then, the point $z = 0$ is a pole of order 2, and the point $z = 1$ is a pole of order 1 (i.e. simple pole). Both the poles lie in the interior enclosed by the contour $C = \{z \in \mathbb{C} : |z| = 2\}$.

$$\text{Res}\,(f,\ 0) = \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz}\left\{\frac{z^2(z^2+3z+2)}{z^2(z-1)}\right\} = -5\ .$$

$$\text{Res}\,(f,\ 1) = \lim_{z\to 1} \frac{(z-1)(z^2+3z+2)}{z^2(z-1)} = 6\ .$$

   By the Cauchy's residue theorem, we have

$$\int_C \frac{(z^2+3z+2)}{(z^3-z^2)}\, dz = 2\pi i\,[\text{Res}\,(f,\ 0) + \text{Res}\,(f,\ 1)] = 2\pi i\ [-5+6] = 2\pi i\ .$$

3. Using the argument principle, evaluate $\dfrac{1}{2\pi i}\displaystyle\int_C \cot z\, dz$, where $C = \{z \in \mathbb{C} : |z| = 7\}$.

   Answer: The function $f(z) = \sin z$ has zeros at $n\pi$, where $n \in \mathbb{Z}$.
   By the argument principle,

$$\int_C \cot z\, dz = \int_C \frac{f'(z)}{f(z)}\, dz = 2\pi i\,(Z - P) = (2\pi i)(5-0) = 10\pi i\ .$$

4. Let $f(z) = (z^3+2)/z$ and $C = \{z(t) = 2e^{it}\,, 0 \le t \le 2\pi\}$. Let Γ denote the image curve under the mapping $w = f(z)$ as z traverses C once. Determine the change in the argument of $f(z)$ as z describes C once. How many times does Γ wind around the origin in the w-plane, and what is the orientation of Γ?

Answer: The function $f(z) = (z^3 + 2)/z$ has a simple pole at $z = 0$ and has three zeros in $|z| < 2$.

$$\Delta_C \arg\,(f(z)) = 2\pi(Z - P)$$

where Z is the number of zeros and P is the number of poles of f in the interior enclosed by C (counting to its multiplicities). So,

$$\Delta_C \arg\,(f(z)) = 2\pi(3 - 1) = 4\pi.$$

The image curve Γ winds around the origin two times in the counterclockwise direction in the w-plane.

5. Find the number of roots of the equation $z^9 - 2z^6 + z^2 - 8z - 2 = 0$, which are lying in $|z| < 1$ with the help of Rouché's theorem.

Answer: Take $g(z) = z^9 - 2z^6 + z^2 - 2$, $f(z) = -8z$ then $P(z) = f(z) + g(z)$. On $|z| = 1$

$$|g(z)| = |z^9 - 2z^6 + z^2 - 2| \leq |z|^9 + 2|z|^6 + |z|^2 + 2 \leq 6$$

and

$$|f(z)| = |-8z| = 8|z| = 8.$$

This implies that

$$|g(z)| \leq 6 < 8 = |f(z)|.$$

Since f has only a simple zero at $z = 0$ in the interior enclosed by $|z| = 1$. Now the Rouché's theorem, the function $f + g \equiv P$ has only one root in $|z| < 1$.

6. How many roots of the equation $z^4 - 5z + 1 = 0$ are lying in the disc $|z| < 1$? And how many roots are lying in the annulus $1 < |z| < 2$?

Answer: In the domain $|z| < 1$:
Set $g(z) = z^4 + 1$, $f(z) = -5z$ and $P(z) = z^4 - 5z + 1$.
Observe that

$$\begin{aligned} |g(z)| = |z^4 + 1| &\leq |z|^4 + 1 \leq 2 \quad \text{for } |z| = 1 \\ |f(z)| = |-5z| &= 5|z| = 5 \quad \text{for } |z| = 1 \\ |g(z)| \leq 2 &< 5 = |f(z)| \quad \text{on } |z| = 1 \end{aligned}$$

Since f has only a simple zero at $z = 0$ in the interior enclosed by $|z| = 1$. By the Rouché's theorem the equation $P(z) = 0$ has only one root in $|z| < 1$.

In the domain $|z| < 2$:
Set $g(z) = -5z + 1$, $f(z) = z^4$ and $P(z) = z^4 - 5z + 1$.

Observe that

$$\begin{aligned}|g(z)| = |-5z+1| &\le 5|z|+1 \le 11 \qquad \text{for } |z|=2\\ |f(z)| = |z^4| &= |z|^4 = 16 \qquad \text{for } |z|=2\\ |g(z)| \le 11 &< 16 = |f(z)| \qquad \text{on } |z|=2\end{aligned}$$

By the Rouché's theorem, the function f and $f+g=P$ have the same number of zeros in the interior enclosed by $|z|=2$. Since f has only a zero of order 4 at $z=0$ in the interior enclosed by $|z|=2$, the function $f+g=P$ has four zeros in the interior enclosed by $|z|=2$. Therefore, the equation $P(z)=0$ has four roots in $|z|<2$.

In the domain $1<|z|<2$:
The equation $P(z)=0$ has 4 roots in $|z|<2$, and it has 1 root in $|z|<1$. It is easy to verify that there is no zero on the set $\{z:|z|=1\}$. Therefore, we conclude that the equation $P(z)=0$ has 3 roots in the domain $1<|z|<2$.

7. Use Rouché's theorem to prove the fundamental theorem of algebra.
Answer: If $p(z)=z^n+a_1z^{n-1}+\cdots+a_n$ then write $f(z)=z^n$ and $g(z)=a_0+a_1z+\cdots+a_{n-1}z^{n-1}$. Now choose $R>0$ such that $|f(z)|>|g(z)|$ on the circle $|z|=R$. Note that

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{a_0}{z^n}+\frac{a_1}{z^{n-1}}+\cdots+\frac{a_{n-1}}{z}\right| \to 0$$

as $z\to\infty$. i.e. for $\epsilon=1>0$ there exists $M>0$ such that

$$\left|\frac{g(z)}{f(z)}\right| < 1 \text{ whenever } |z|>M.$$

Take $R=M+1$ and apply Rouché's theorem.

8. Determine the isolated singularities and compute the residue of the functions

$$a)\ \frac{e^z}{z^2-1},\quad b)\ \frac{3z}{z^2+iz+2},\quad c)\ \cot\pi z,\quad d)\ \frac{\pi\cot\pi z}{(z+\frac{1}{2})^2}.$$

Answer:

a) Singularities at ±1. Residue at $1=\lim_{z\to1}(z-1)\dfrac{e^z}{z^2-1}=\dfrac{e}{2}$
Residue at $-1=\lim_{z\to-1}(z+1)\dfrac{e^z}{z^2-1}=-\dfrac{1}{2e}$.

b) Singularities at i and $-2i$.
Residue at $i=\lim_{z\to i}(z-i)\dfrac{3z}{z^2+iz+2}=1$
Residue at $-2i=\lim_{z\to-2i}(z+2i)\dfrac{3z}{z^2+iz+2}=2$.

c) Singularities at $\pm n$.
Residue at $n=\lim_{z\to n}(z-n)\cot\pi z=\lim_{z\to n}(z-n)\dfrac{(-1)^n\cos\pi z}{\sin(z-n)\pi}=\dfrac{1}{\pi}$.

d) Singularities at $\pm n$ and $-\frac{1}{2}$.

Residue at $\pm n = \lim_{z \to \pm n} (z \mp n) \dfrac{\pi \cot \pi z}{(z+\frac{1}{2})^2} = \dfrac{1}{(\pm n + \frac{1}{2})^2}$.

Residue at $-\frac{1}{2} = \lim_{z \to -\frac{1}{2}} \dfrac{1}{2} \dfrac{d}{dz} \left[\left(z + \dfrac{1}{2} \right)^2 \dfrac{\pi \cot \pi z}{(z+\frac{1}{2})^2} \right] = -\dfrac{\pi^2}{2}$.

Questions Q9-Q13: do yourself.

9. State where the following mappings are conformal.

   (i) $w = \sin z$ (ii) $w = z^2 + 2z$.

10. Show that the mapping $w = \cos z$ is not conformal at $z_0 = 0$.

11. Find a bilinear transformation which maps:

    (i) 2, i, -2 onto 1, i, -1. (ii) i, -1, 1 onto 0, 1, ∞ (iii) ∞, i, 0 onto 0, i, ∞

12. Show that the transformation $w = \dfrac{z - i}{1 - iz}$ maps the interior enclosed by the circle $|z| = 1$ onto the lower half-plane $\mathrm{Im}\,(w) < 0$.

13. Find the image of the straight line $\mathrm{Re}\,(z) = a$ (constant) in the z-plane under the mapping $w = \dfrac{z-1}{z+1}$.