MA201: Complex Analysis

Assignment 7

(Residue theorem, argument principle, Rouche's theorem and Mobius transforms)

July - November 2024

(1) Evaluate $\displaystyle\int_{|z-\frac{\pi}{2}|=\frac{\pi}{2}} \frac{z}{\cos z}\, dz$.

(2) Using Cauchy's residue theorem, evaluate $\displaystyle\int_{|z|=2} \frac{(z^2+3z+2)}{(z^3-z^2)}\, dz$.

(3) Using the argument principle, evaluate $\displaystyle\frac{1}{2\pi i}\int_C \cot z\, dz$, where $C = \{z \in \mathbb{C} : |z| = 7\}$.

(4) Let $f(z) = (z^3+2)/z$ and $C = \{z(t) = 2e^{it}, 0 \le t \le 2\pi\}$. Let Γ denote the image curve under the mapping $w = f(z)$ as z traverses C once. Determine the change in the argument of $f(z)$ as z describes C once. How many times does Γ wind around the origin in the w-plane, and what is the orientation of Γ?

(5) Find the number of roots of the equation $z^9 - 2z^6 + z^2 - 8z - 2 = 0$, which are lying in $|z| < 1$ with the help of Rouché's theorem.

(6) How many roots of the equation $z^4 - 5z + 1 = 0$ are lying in the disc $|z| < 1$? And how many roots are lying in the annulus $1 < |z| < 2$?

(7) Use Rouché's theorem to prove the fundamental theorem of algebra.

(8) Determine the isolated singularities and compute the residue of the functions

$$a)\ \frac{e^z}{z^2-1}, \quad b)\ \frac{3z}{z^2+iz+2}, \quad c)\ \cot \pi z, \quad d)\ \frac{\pi \cot \pi z}{(z+\frac{1}{2})^2}.$$

(9) State where the following mappings are conformal.

(i) $w = \sin z$ (ii) $w = z^2 + 2z$.

(10) Show that the mapping $w = \cos z$ is not conformal at $z_0 = 0$.

(11) Find a bilinear transformation which maps:

(i) $2, i, -2$ onto $1, i, -1$. (ii) $i, -1, 1$ onto $0, 1, \infty$ (iii) $\infty, i, 0$ onto $0, i, \infty$

(12) Show that the transformation $w = \dfrac{z-i}{1-iz}$ maps the interior of the circle $|z| = 1$ onto the lower half-plane $\operatorname{Im}(w) < 0$.

(13) Find the image of the straight line $\operatorname{Re}(z) = a$ (constant) in the z-plane under the mapping $w = \dfrac{z-1}{z+1}$.