Applications of Cauchy's Integral Formula

Fundamental Theorem of Algebra

Fundamental Theorem of Algebra (FTA): Every polynomial $p(z)$ of degree $n \geq 1$ has a root in the complex plane $\mathbb{C}$.

Proof: Suppose $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ is a polynomial with no root in $\mathbb{C}$. Then $\frac{1}{P(z)}$ is an entire function. Since

$$\left|\frac{P(z)}{z^n}\right| = \left|1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right| \to 1, \text{ as } |z| \to \infty,$$

it follows that $|p(z)| \to \infty$ and hence $|1/p(z)| \to 0$ as $|z| \to \infty$. Consequently, $\frac{1}{p(z)}$ is a bounded function. Hence by Liouville's theorem $\frac{1}{p(z)}$ is constant, which is not possible.

Exercise: Show that every polynomial $P(z)$ of degree n has exactly n many roots in the complex plane $\mathbb{C}$.

Morera's Theorem

Morera's Theorem: If f is continuous in a simply connected domain D, and if

$$\int_C f(z)dz = 0$$

for every simple closed contour C in D, then f is analytic.

Proof: Fix a point $z_0 \in D$, and define

$$F(z) = \int_{z_0}^{z} f(w)dw.$$

Use the idea of proof of the existence of antiderivative to show that $F' = f$. That is, F is complex differentiable. Since f is continuous, F is analytic. Now, by Cauchy integral formula f is analytic.

Result: Let f be a continuous function on an open set D and γ be a simple closed curve in $D \subset \mathbb{C}$. Then, for each $\epsilon > 0$, there exists a polygon P such that

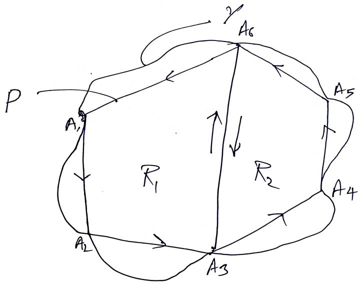
$$\left| \int_\gamma f(z)dz - \int_P f(z)dz \right| < \epsilon.$$

Proof: Refer to Lemma 1.19, page 65, Functions of one complex variable by J. B. Conway.

Morera's Theorem

Corollary to Morera's Theorem: If f is continuous in a simply connected domain D, and if $\int_{\partial R} f(z)dz = 0$ for every rectangle R in D, then f is analytic.

Proof: By the above result, for each $\epsilon > 0$, there exists a polygon P such that $|\int_{\gamma} f(z)dz - \int_{P} f(z)dz| < \epsilon$. Now, $\int_{P} f(z)dz = \sum_{j=1}^{n} \int_{\partial R_j} f(z)dz = 0$. Hence $|\int_{\gamma} f(z)dz| < \epsilon$ which is true for each $\epsilon > 0$. Thus, $\int_{\gamma} f(z)dz = 0$. The analyticity of f follows from Morera's theorem.



Application of Morera's Theorem

Question: Let $\{f_n\}$, be a sequence of analytic functions converging uniformly to a continuous function f on the open disc $B(0,1)$. Is f is analytic in $B(0,1)$?

By Cauchy's theorem, we know that

$$\int_{\gamma} f_n(z)\,dz = 0$$

$\forall n \in \mathbb{N}$ and for any closed curve γ in the disc $B(0,1)$. Then

$$\int_{\gamma} f(z)\,dz = \int_{\gamma} \lim_{n\to\infty} f_n(z)\,dz = \lim_{n\to\infty} \int_{\gamma} f_n(z)\,dz = 0.$$

for every closed curve γ. By Morera's theorem f must be analytic.

Application of Morera's Theorem

Question: Let $h : [a, b] \times B(0, 1) \to \mathbb{C}$ be continuous. If for each fixed t, $h(t, z)$ is analytic function of z in $B(0, 1)$, then

$$H(z) = \int_a^b h(t, z)dt$$

is analytic on $B(0, 1)$.

Proof:

- Notice that if $z_n \to z$, then $h(t, z_n) \to h(t, z)$ uniformly for $a \leq t \leq b$.
- That means $H(z_n) \to H(z)$. So H is continuous in $B(0, 1)$.
- Let γ be any simple closed curve in $B(0, 1)$. So by Cauchy's theorem

$$\int_\gamma h(t, z)dz = 0.$$

- Then by Fubini's theorem, we get

$$\int_\gamma H(z)dz = \int_\gamma \left(\int_a^b h(t, z)dt \right) dz = \int_a^b \int_\gamma h(t, z)dzdt = 0.$$

- By Morera's theorem, H is analytic in $B(0, 1)$.

Application of Morera's Theorem

Question: Let D be a domain and let $f : D \to \mathbb{C}$ be continuous. If f is analytic function of z in $D \setminus \mathbb{R}$, then f is analytic on D.

Proof: Let R be a closed rectangle contained in D and ∂R be the boundary of closed rectangle R.

- **Case I:** R does not meet the real axis. Then $\int_{\partial R} f(z)dz = 0$ by Cauchy's theorem.
- **Case II:** One edge of R lies on the real axis. For given $\epsilon > 0$ let $R_\epsilon = \{z \in R : \mathrm{Im(z)} \geq \epsilon\}$. In this case,

$$\int_{\partial R} f(z)dz = \lim_{\epsilon \to 0} \int_{\partial R_\epsilon} f(z)dz.$$

  Indeed, the integral along the bottom edge has the form $\int_a^b f(t + i\epsilon)dt$ and converges to $\int_a^b f(t)dt$ as $\epsilon \to 0$ (as $f(t + i\epsilon) \to f(t)$ uniformly for $a \leq t \leq b$.)

- **Case III:** The top edge of R is in the upper half plane, and the bottom edge of R is in the lower half plane.
  Let R_+ be the part of R in the closed UHP and R_- be the part of R in the LHP. Then

$$\int_{\partial R} f(z)dz = \int_{\partial R_+} f(z)dz + \int_{\partial R_-} f(z)dz = 0,$$

  by the previous case, the analyticity of f follows from Morera's theorem.

  Exercise: Let L be a line in the complex plane. If f is continuous on a domain D that is analytic on $D \setminus L$, then show that f is analytic on D.