Power Series

Series of complex numbers

For a sequence of complex numbers z_n in $\mathbb{C}$, the series $\sum_{n=0}^{\infty} z_n$ converges

- to z, if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\left| \sum_{n=0}^{m} z_n - z \right| < \epsilon$, whenever $m \geq N$;
- **absolutely** if $\sum_{n=0}^{\infty} |z_n|$ converges.
- If the series $\sum_{n=0}^{\infty} z_n$ converges absolutely, then $\sum_{n=0}^{\infty} z_n$ converges.
- Let $S_N = \sum_{n=0}^{N} z_n$ be the Nth partial sum of $\sum_{n=0}^{\infty} z_n$. Then, the series $\sum_{n=0}^{\infty} z_n$ converges if and only if the sequence $\{S_N\}$ converges.

Power Series

- Definition: A series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where $a_n \in \mathbb{C}$ and $z_0 \in \mathbb{C}$ is called a **power series** around the point z_0.
- For what values of z, do the following power series converge?
  1. $\sum_{n=0}^{\infty} z^n$ ($|z| < 1$, geometric series.)
  2. $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ (for all z, exponential series.)
  3. $\sum_{n=0}^{\infty} n! z^n$, (only $z = 0$)

Power Series

- If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \in \mathbb{C}$, then it converges for all $z \in \mathbb{C}$ with $|z| < |z_0|$.
- **Proof.** It follows from the hypothesis that there exists $M \geq 0$ such that $|a_n z_0^n| \leq M$ for all $n \in \mathbb{N}$.
- Note that
$$|a_n z^n| = |a_n z_0|^n \left| \frac{z}{z_0} \right|^n \leq M \left| \frac{z}{z_0} \right|^n .$$
- The proof will be followed by the comparison test and the behavior of the geometric series.

Power Series

- (**Radius of convergence**) Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there always exists $0 \leq R \leq \infty$ such that:
  1. If $|z| < R$, the series converges absolutely.
  2. If $|z| > R$, the series diverges.

  The number R is called the **radius of convergence** of the power series.
- (a) For $\sum_{n=0}^{\infty} n! z^n$, $R = 0$. (b) For $\sum_{n=0}^{\infty} z^n$, $R = 1$. (c) For $\sum_{n=1}^{\infty} \frac{z^n}{n}$, $R = 1$. (d) For $\sum_{n=1}^{\infty} \frac{z^n}{n!}$, $R = \infty$.
- **Remark:** Note that no conclusion about convergence can be drawn if $|z| = R$. The power series in (c) the above does not converge if $z = 1$ but converges if $z = -1$.

Question: Why can't the power series converge at every point in $|z| = R$?

- The formula for calculating R goes precisely as in the case of reals, that is,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|},$$

whenever the above limits exist (with the adaptation that division by ∞ (resp. 0) produces 0 (resp. ∞)).

- Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Then, for all $z \in B(0, R)$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a well defined function.

- **Question:** Is f analytic on $B(0, R)$?

Theorem: Suppose $F(z) = \sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence $R > 0$.

Then

1. the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has the same radius of convergence R.
2. the function F is differentiable on $B(0, R)$ and $F'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

Proof. Since we know $\lim_{n\to\infty} n^{1/n} = 1$, therefore,

$$\limsup |a_n|^{1/n} = \limsup |n a_n|^{1/n}.$$

Hence $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

Now, let $|z| < r < R$, and write

$$F(z) = S_N(z) + R_N(z),$$

where

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \quad \text{and} \quad R_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Let us denote

$$f(z) = \sum_{n=1}^{\infty} a_n \, n z^{n-1}.$$

Then, if h is chosen so that $|z + h| < r$, we have

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} - f(z) \quad &= \quad \left(\frac{S_N(z+h) - S_N(z)}{h} - S_N'(z) \right) \\ &+ \quad (S_N'(z) - f(z)) + \left(\frac{R_N(z+h) - R_N(z)}{h} \right). \end{aligned}$$

Next, we will show that all three expressions on the right hand-side in the above equation will go to zero for large N and small $|h|$.

Since $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + ab^{n-1})$, we can write

$$\left|\frac{R_N(z+h) - R_N(z)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z+h)^n - z^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1},$$

where we have used the fact that $|z| < r$ and $|z+h| < r$. Since the right most expression is the reminder (or tail) of a convergent series. Given $\epsilon > 0$, we can find $N_1 \in \mathbb{N}$ such that

$$\left|\frac{R_N(z+h) - R_N(z)}{h}\right| < \frac{\epsilon}{3}, \tag{1}$$

whenever $N > N_1$. Also $\lim_{N\to\infty} S_N'(z) = f(z)$, we can find N_2 so that $N > N_2$ implies that

$$|S_N'(z) - f(z)| < \frac{\epsilon}{3}. \tag{2}$$

If $N > \max\{N_1, N_2\}$, then both (1) and (2) will hold together.

Note that $S_N(z)$ is a polynomial, and the derivative of a polynomial is obtained by differentiating it term by term. Given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$\left| \frac{S_N(z+h) - S_N(z)}{h} - S_N'(z) \right| < \frac{\epsilon}{3}, \tag{3}$$

whenever $|h| < \delta$.

By combining (1), (2) and (3), we get

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| < \epsilon,$$

whenever $|h| < \delta$.

Corollary: The function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is infinitely differentiable on $B(0, R)$ and the higher derivatives are also power series obtained via term-by-term differentiation and have the same radius of convergence R.

Theorem: Suppose $\sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence $R > 0$. If $0 < r < R$, then the above series converges uniformly on $\overline{B(0,r)}$.

Proof. If $r < \rho < R$, then $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{R} < \frac{1}{\rho}$. Hence, there exists $N \in \mathbb{N}$ such that $|a_n| < \frac{1}{\rho^n}$ for all $n > N$. Now, if $|z| \leq r$, then

$$|a_n z^n| < \left(\frac{r}{\rho}\right)^n,$$

whenever $n > N$. This implies

$$\sum_{n=N+1}^{\infty} |a_n z^n| \leq \sum_{n=N+1}^{\infty} \left(\frac{r}{\rho}\right)^n.$$

Therefore, the series is uniformly convergent by the Weierstrass M-test.