Taylor's Theorem

Question: Let $f : B(z_0, R) \to \mathbb{C}$ be analytic. Can f be represented as a power series around z_0?

(**Taylor's Theorem:**) Let f be analytic on $D = B(z_0, R)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{for all } z \in D,$$

where $a_n = \dfrac{f^n(z_0)}{n!}$; for $n = 0, 1, 2, \ldots$.

Proof. Without loss of generality, we consider $z_0 = 0$. Fix $z \in B(0, R)$, and let $|z| = r$. Let C_0 be a circle centered at 0 and having radius r_0 such that $r < r_0 < R$. Note that for any distinct pair of complex numbers, z and w satisfies

$$\frac{1}{w - z} = \frac{1}{w} + \frac{z}{w^2} + \frac{z^2}{w^3} + \ldots + \frac{z^{n-1}}{w^n} + \frac{z^n}{(w - z)w^n}. \text{ (\textit{check}!)}$$

Taylor Series

By Cauchy's integral formula, we now have

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{|w|=r_0} \frac{f(w)dw}{w-z} \\
&= \frac{1}{2\pi i} \int_{|w|=r_0} f(w) \left[\frac{1}{w} + \frac{z}{w^2} + \ldots + \frac{z^{n-1}}{w^n} \right] dw + \frac{z^n}{2\pi i} \int_{|w|=r_0} \frac{f(w)dw}{(w-z)w^n} \\
&= f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \ldots\ldots + \frac{f^{n-1}(0)}{(n-1)!} z^{n-1} + \rho_n(z) \\
&= \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} z^k + \rho_n(z),
\end{aligned}
$$

where $\rho_n(z) = \frac{z^n}{2\pi i} \int_{C_0} \frac{f(w)dw}{(w-z)w^n}$.

Taylor Series

Now, it is enough to show that $\lim_{n\to\infty} |\rho_n(z)| = 0$. Notice that the function $w \to \dfrac{f(w)}{w-z}$ is bounded on the circle C_0 (as it is continuous). If $\left|\dfrac{f(w)}{w-z}\right| \leq M$ for all $w \in C_0$, then by ML inequality, it follows that

$$|\rho_n(z)| \leq Mr_0 \left|\frac{z}{r_0}\right|^n.$$

Since $|z| = r < r_0$ and $|w| = r_0$ implies $\lim_{n\to\infty} |\rho_n(z)| = 0$ as $n \to \infty$.

Remark: If f is an entire function, then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{for all } z \in \mathbb{C},$$

where $a_n = \dfrac{f^n(0)}{n!}$; for $n = 0, 1, 2, \ldots$.

Exponential function

The power series $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ has the radius of convergence ∞. If we define $f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, then f is an entire function.

1. $f'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = f(z)$.
2. We know that $\frac{d}{dz} e^z = e^z$. Is $f(z) = e^z$?
3. **Yes.** If $h(z) = \frac{f(z)}{e^z}$, then $h'(z) = 0$ for all $z \in \mathbb{C}$. Therefore, $f(z) = ce^z$. But $f(0) = 1$. Implies $c = 1$.

Exponential function

Now, we can write exponential function as a power series

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The $\sin z$ and $\cos z$ functions can also be written as a power series by using the exponential series.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

and similarly,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}.$$

Exponential function

Euler's Formula:

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{(2n)!}(i\theta)^{2n} + \frac{1}{(2n+1)!}(i\theta)^{2n+1} \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{\theta^{2n}(i^2)^n}{(2n!)} + i\frac{\theta^{2n+1}(i^2)^n}{(2n+1)!} \right] \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

Polynomial function

If $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a polynomial in $\mathbb{C}$ of degree $n \geq 1$, then $|p(z)| \leq M|z|^n$ while $|z| > 1$.

Question: Let f is an entire function such that $|f(z)| \leq M|z|^n$ for $|z| > 1$. Does it imply that the function f is a polynomial?

Answer: Yes!

By Cauchy's estimate,

$$|f^k(0)| \leq \frac{k!MR^n}{R^k} = \frac{k!M}{R^{(k-n)}} \to 0$$

as $R \to \infty$ for each $k > n$. Since f is an entire function, therefore, $f(z) = \sum_{k=1}^{\infty} a_k z^k$. By the above observation $a_k = 0$ for all $k > n$. That is, f is a polynomial of degree at most n.

Remark: The condition $|z| > 1$ can be replaced with $|z| > \delta$ for any $\delta \geq 0$. Therefore, it is enough to consider that $|f(z)| \leq M|z|^n$ while $|z| \to \infty$.