Zeros of analytic functions

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Suppose that $f : D \to \mathbb{C}$ is an analytic function on an open set $D \subseteq \mathbb{C}$.

- A point $z_0 \in D$ is called zero of f if $f(z_0) = 0$.
- The z_0 is a zero of multiplicity/order m, if there is an analytic function $g : D \to \mathbb{C}$ such that

  $$f(z) = (z - z_0)^m g(z), \ \ g(z_0) \neq 0.$$

  In this case, we have $f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ but $f^m(z_0) \neq 0$.
- **Understanding of multiplicity via Taylor's series:** If f is an analytic function in D, then f has a Taylor series expansion around z_0 as

  $$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n, \quad |z - z_0| < R.$$

  If f has a zero of order m at z_0, then

  $$f(z) = (z - z_0)^m \sum_{n=m}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^{n-m}$$

  and define $g(z) = \sum_{n=m}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^{n-m}$.

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Zeros of a non-constant analytic function are isolated: If $f : D \to \mathbb{C}$ is non-constant and analytic at $z_0 \in D$ with $f(z_0) = 0$, then there is an $R > 0$ such that $f(z) \neq 0$ for $z \in B(z_0, R) \setminus \{z_0\}$.

Proof. Assume that f has a zero at z_0 of order m. Then

$$f(z) = (z - z_0)^m g(z),$$

where $g(z)$ is analytic and $g(z_0) \neq 0$. For $\epsilon = \dfrac{|g(z_0)|}{2} > 0$, we can find a $\delta > 0$ such that

$$|g(z) - g(z_0)| < \frac{|g(z_0)|}{2},$$

whenever $|z - z_0| < \delta$ (as g is continuous at z_0). Therefore, whenever $|z - z_0| < \delta$, we have $0 < \dfrac{|g(z_0)|}{2} < |g(z)| < \dfrac{3|g(z_0)|}{2}$. Take $R = \delta$.

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Identity Theorem: Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be analytic. If there exists an infinite sequence $\{z_k\} \subset D$ such that $f(z_k) = 0\ \forall k \in \mathbb{N}$ and $z_k \to z_0 \in D$, then $f(z) = 0$ for all $z \in D$.

Proof. Case I: If $D = \{z \in \mathbb{C} : |z - z_0| < r\}$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ for all } z \in D.$$

To show $f \equiv 0$ on D, it is enough to show $f^n(z_0) = 0$ for all n. If possible, assume that $f^n(z_0) \neq 0$ for some $n > 0$. Let n_0 be the smallest positive integer such that $f^{n_0}(z_0) \neq 0$. Then

$$f(z) = \sum_{n=n_0}^{\infty} a_n (z - z_0)^n = (z - z_0)^{n_0} g(z),$$

where $g(z_0) = a_{n_0} \neq 0$. Since g is continuous at z_0, there exists $\delta > 0$ such that $g(z) \neq 0$ for all $z \in B(z_0, \delta)$. By hypothesis, there exists some k such that $z_0 \neq z_k \in B(z_0, \delta)$ and $f(z_k) = 0$. This forces $g(z_k) = 0$, which is a contradiction.

Case II: If D is a domain.

- Since $z_0 \in D$, therefore, there exists $\delta > 0$ such that $B(z_0, \delta) \subset D$.
- By Case I, $f(z) = 0, \ \forall\ z \in B(z_0, \delta)$.
- Take $z \in D$ and join z and z_0 by a line segment. Cover the line segments with open balls so that the center of the ball lies in the previous ball. Apply the above argument to get $f(z) = 0$ for all $z \in D$.

Uniqueness Theorem: Let $D \subseteq \mathbb{C}$ be a domain and $f, g : D \to \mathbb{C}$ is analytic. If there exists an infinite sequence $\{z_n\} \subset D$ such that $f(z_n) = g(z_n), \ \forall n \in \mathbb{N}$ and $z_n \to z_0 \in D$, then $f(z) = g(z)$ for all $z \in D$.

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- Find all entire functions f such that $f(x) = \cos x + i \sin x$ for all $x \in (0, 1)$.
- Find all entire functions f such that $f(r) = 0$ for all $r \in Q$.
- Find all analytic functions $f : B(0, 1) \to \mathbb{C}$ such that $f(\frac{1}{n}) = \sin(\frac{1}{n}), \ \forall n \in \mathbb{N}$.
- There does not exists any analytic function f defined on $B(0, 1)$ such that $f(x) = |x|^3$. (Hint: For the sequence $\dfrac{1}{n} \to 0 \in \mathbb{C}$, the function f satisfying $f(\dfrac{1}{n}) = \dfrac{1}{n^3}$. Hence by the uniqueness theorem $f(z) = z^3$. On the other hand, for the sequence $\dfrac{i}{n} \to 0 \in \mathbb{C}$, the function f satisfying $f(\dfrac{1}{n}) = \dfrac{-i}{n^3}$. So $f(z) = -i\, z^3$, which is impossible.)