Maximum Modulus Theorem and Laurent Series

Maximum Modulus Theorem: Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be analytic, and if there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$, $\forall z \in D$, then f is constant on D.

Proof. Choose an $r > 0$ such that $\overline{B(z_0, r)} \subset D$. Let $\gamma(t) = z_0 + \rho e^{it}$ for $0 \leq t \leq 2\pi$ and $0 < \rho < r$. By Cauchy's integral formula, we get

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it})\, dt.$$

Hence

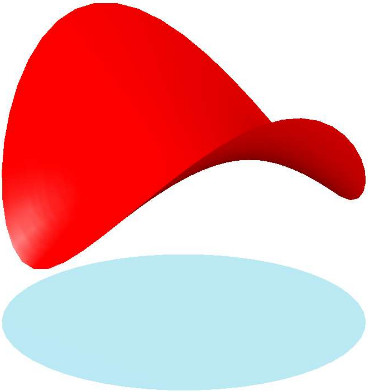
$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})|\, dt \leq |f(z_0)|.$$

This gives

$$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0 + \rho e^{it})| \right] dt = 0.$$

It follows that $|f(z_0)| = |f(z_0 + \rho e^{it})|$ for all t and ρ. Hence f is constant on $B(z_0, r)$. By applying the identity theorem, we conclude that f is constant throughout the domain D.

Maximum Modulus Theorem



A plot of the modulus of $\cos z$ (shown in red) in the unit disk centered at the origin (shown in blue).

Consequences

- If f is analytic in a bounded domain D and continuous on ∂D, then $|f(z)|$ attains its maximum at some point on the boundary ∂D.
- Consider the function $f(z) = e^{e^z}$ for $z \in D = \{z \in \mathbb{C} : |\text{Im } z| < \frac{\pi}{2}\}$. Then for $a \pm i\frac{\pi}{2} \in \partial D = \{\zeta \in \mathbb{C} : |\text{Im } \zeta| = \frac{\pi}{2}\}$,

  $$f(a \pm i\frac{\pi}{2}) = \left| e^{e^{a \pm i\frac{\pi}{2}}} \right| = \left| e^{\pm i e^a} \right| = 1.$$

  Again, if $x \in \mathbb{R} \subset D$ then, $f(x) = e^{e^x} \to \infty$ as $x \to \infty$.
- **Minimum Modulus Theorem** Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ be analytic, and if there exists a point $z_0 \in D$ such that $|f(z)| \geq |f(z_0)|$ for all $z \in D$, then either f is a constant function or $f(z_0) = 0$.

Hint. Apply maximum modulus theorem on $1/f$.

Exercise: Let f be an analytic function on $D = \{z \in \mathbb{C} : |z| > R\}$. Suppose there exists $M > 0$ such that $|f(z)| \leq M$ on ∂D. If $\lim_{|z| \to \infty} |f(z)| \leq M$, then show that $|f(z)| \leq M$ on D. (Hint: Use maximum modulus theorem.)

Laurent's Theorem

Suppose that $0 \leq r < R$. Let f be an analytic defined on the annulus

$$A = \text{ann}(a, r, R) = \{z : r < |z - a| < R\}.$$

Then for each $z \in A$, $f(z)$ has the Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n$$

where the convergence is absolute and uniform in $\overline{\text{ann}(a, r_1, R_1)}$ if $r < r_1 < R_1 < R$. The coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - a)^{n+1}}\, dz,$$

where $\gamma(t) = a + se^{it}, \ t \in [0, 2\pi]$ and for any $r < s < R$.

Moreover, this series is unique and $\sum_{k=-\infty}^{-1} a_n (z - a)^n$ is called **(principal part)**

and $\sum_{k=0}^{\infty} a_n (z - a)^n$ is called **(regular/analytic part)**.

Examples of Laurent Series

Let $f(z) = \dfrac{1}{1-z}$.

- On the domain $|z| < 1$

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots + .$$

- On the domain $|z| > 1$ i.e. $\frac{1}{|z|} < 1$, by the above mentioned fact we have

$$f(z) = \frac{1}{1-z} = \frac{-1}{z(1-\frac{1}{z})} = -\frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = -\frac{1}{z} - \frac{1}{z^2} - \ldots .$$

Examples of Laurent Series

- Let $f(z) = \dfrac{\sin z}{z}$ domain $|z| > 0$

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{6} + \frac{z^4}{120} + \cdots.$$

- Let $f(z) = \dfrac{e^z - 1}{z^3}$ domain $|z| > 0$

$$f(z) = \frac{e^z - 1}{z^3} = \frac{1}{z^3}\sum_{1}^{\infty}\frac{z^n}{n!} = \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + \frac{z}{24} + \cdots.$$

- Let $f(z) = e^{\frac{1}{z}}$ domain $|z| > 0$

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots.$$

Examples of Laurent Series

Let $f(z) = \dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$

- Domain $|z| < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} (1 - 2^{-n-1}) z^n.$$

- Domain $1 < |z| < 2$ i.e. $\dfrac{1}{|z|} < 1$ and $\dfrac{|z|}{2} < 1$, so we have

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

- Domain $|z| > 2$ i.e. $\dfrac{1}{|z|} < 1$ and $\dfrac{2}{|z|} < 1$, in this case we have

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} - \sum_{n=1}^{\infty} \frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{2^{n-1} - 1}{z^n}.$$

Problems

- Let f be an entire function such that $\lim_{z\to\infty}\left|\frac{f(z)}{z}\right| = 0$. Show that f is constant.