Singularities

Singularities

What can we say about the behavior of the following functions at 0?

- $f(z) = \dfrac{1}{z^9}$
- $f(z) = \dfrac{\sin z}{z}$
- $f(z) = \dfrac{e^z - 1}{z}$
- $f(z) = \dfrac{1}{\sin(\frac{1}{z})}$
- $f(z) = \text{Log } z$
- $f(z) = e^{\frac{1}{z}}$

In the above, we observe that none of the functions is analytic at 0; however, in each neighborhood of 0 contains a point at which these functions are analytic.

Definition: The point z_0 is called a **singular point** or **singularity of f** if f is not analytic at z_0, but each neighborhood of z_0 contains at least one point at which f is analytic.

- Here $\dfrac{e^z - 1}{z}, \dfrac{1}{z^2}, \sin\frac{1}{z}, \text{Log } z$, etc. has singularity at $z = 0$.
- $\bar{z}, |z|^2, \text{Re } z, \text{Im } z, z\text{Re } z$ are nowhere analytic. That does not mean that every point of $\mathbb{C}$ is a singularity.
- A singularity can be classified into TWO ways:
  1. A singular point z_0 is said to be an **isolated singularity or isolated singular point** of f if f is analytic in $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$.
  2. A singular point z_0 is said to be a **non-isolated singularity** if z_0 is not an isolated singular point.
- $\dfrac{\sin z}{z}, \dfrac{1}{z^2}, \sin(\frac{1}{z})$ (0 is an isolated singular point).
- $\dfrac{1}{\sin(\pi/z)}$, *Log* z these functions have non-isolated singularity at 0.

Singularities

If f has an isolated singularity at z_0, then f is analytic in $B(z_0, r) \setminus \{z_0\}$ for some small $r > 0$. In this case, f has the following Laurent series expansion:

$$f(z) = \cdots \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - a) + a_2(z - z_0)^2 + \cdots .$$

- If all $a_{-n} = 0$ for all $n \in \mathbb{N}$, then the point $z = z_0$ is a **removable singularity**.
- The point $z = z_0$ is called a **pole** if all but a finitely many of a_{-n}'s are non-zero. If m is the highest integer such that $a_{-m} \neq 0$, then z_0 is a pole of order m.
- If $a_{-n} \neq 0$ for infinitely many n's, then the point $z = z_0$ is an **essential singularity**.
- The term a_{-1} is called the **residue** of f at z_0.

Removable singularities

- The following statements are equivalent:
  1. f has a removable singularity at z_0.
  2. If all $a_{-n} = 0$ for all $n \in \mathbb{N}$.
  3. $\lim_{z \to z_0} f(z)$ exists and is finite.
  4. $\lim_{z \to z_0} (z - z_0) f(z) = 0$.
  5. f is bounded in a deleted neighborhood of z_0.
- The function $\dfrac{\sin z}{z}$ has removable singularity at 0.

Pole

The following statements are equivalent:

- f has a pole of at z_0.
- $\lim_{z \to z_0} |f(z)| = \infty$.

The following statements are equivalent:

- f has a pole of order m at z_0.
- $f(z) = \dfrac{g(z)}{(z - z_0)^m}$, g is analytic at z_0 and $g(z_0) \neq 0$.
- $\dfrac{1}{f}$ has a zero of order m.
- $\lim_{z \to z_0} (z - z_0)^{m+1} f(z) = 0$
- $\lim_{z \to z_0} (z - z_0)^m f(z)$ has removable singularity at z_0.

Essential singularity

The following statements are equivalent:

- f has an essential singularity at z_0.
- The point z_0 is neither a pole nor a removable singularity of f.
- $\lim_{z \to z_0} f(z)$ does not exist.
- There are infinitely many terms in the principal part of the Laurent series expansion of f around the point z_0.

The limit point of zeros is an isolated essential singularity if the limit is itself a singular point. For example:

$$f(z) = \sin \frac{1}{z}$$

Singularities at ∞

Let f be a complex valued function. Define another function g by

$$g(z) = f\left(\frac{1}{z}\right).$$

Then the nature of the singularity of f at $z = \infty$ is defined as the nature of the singularity of g at $z = 0$.

- $f(z) = z^3$ has a pole of order 3 at ∞.
- e^z has an essential singularity at ∞.
- An entire function f has a removable singularity at ∞ if and only if f is constant.(Prove This!)
- An entire function f has a pole of order m at ∞ if and only if f is a polynomial of degree m.(Prove This!)