Evaluation of integrals - I

Evaluation of certain contour integrals: type I

Type I: Consider the integrals of the form

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta.$$

- If we take $z = e^{i\theta}$, then $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$ and $d\theta = \dfrac{dz}{iz}$.
- Substituting for $\sin\theta, \cos\theta$ and $d\theta$ the definite integral transforms into the following contour integral

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_{|z|=1} f(z)\, dz,$$

  where $f(z) = \frac{1}{iz}[F(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z}))]$
- Apply the residue theorem to the integral

$$\int_{|z|=1} f(z)\, dz.$$

Example of type I

Consider

$$\int_0^{2\pi} \frac{1}{1+3(\cos t)^2}\, dt.$$

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1+3(\cos t)^2}\, dt &= \int_{|z|=1} \frac{1}{1+3(\frac{1}{2}(z+\frac{1}{z}))^2} \frac{dz}{iz} \\
&= -4i \int_{|z|=1} \frac{z}{3z^4+10z^2+3}\, dz \\
&= -4i \int_{|z|=1} \frac{z}{3(z+\sqrt{3}i)\left(z-\sqrt{3}i\right)\left(z+\frac{i}{\sqrt{3}}\right)\left(z-\frac{i}{\sqrt{3}}\right)}\, dz \\
&= -\frac{4}{3} i \int_{|z|=1} \frac{z}{(z+\sqrt{3}i)(z-\sqrt{3}i)\left(z+\frac{i}{\sqrt{3}}\right)\left(z-\frac{i}{\sqrt{3}}\right)}\, dz \\
&= -\frac{4}{3} i \times 2\pi i \{Res(f, \frac{i}{\sqrt{3}}) + Res(f, -\frac{i}{\sqrt{3}})\}.
\end{aligned}$$

Improper integrals of rational functions

- A function f on $[0, \infty)$ is said to be improperly integrable if $\int_0^b f(x)dx$ exists for each $b > 0$, and $\lim_{b\to\infty} \int_0^b f(x)dx$ exists. In this case, we write

$$\int_0^\infty f(x)dx := \lim_{b\to\infty} \int_0^b f(x)dx.$$

- If f is defined for all real x, then the improper integral of f over $(-\infty, \infty)$ is defined by

$$\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty} \int_a^0 f(x)dx + \lim_{b\to\infty} \int_0^b f(x)dx$$

provided both limits exist.

- There is another value associated with $\int_{-\infty}^\infty f(x)dx$, namely the **Cauchy's principal value(P.V.)**, and it is given by

$$\text{P. V.} \int_{-\infty}^\infty f(x)dx := \lim_{R\to\infty} \int_{-R}^R f(x)dx$$

provided the limit exists.

Evaluation of certain contour integrals: type II

- If the function f is improperly integrable on $(-\infty, \infty)$, Cauchy's principle value integral exists and is equal to improper integral. That is,

$$\int_{-\infty}^{\infty} f(x)dx = \text{P. V.} \int_{-\infty}^{\infty} f(x)dx.$$

- However, the existence of Cauchy's principle value integral does not imply the existence of the improper integral. Take $f(x) = x$.
- However, if f is an even function (i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}$), then both forms of integral exist and are equal.

Consider the rational function $f(z) = \dfrac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials with real coefficients such that

- $Q(z)$ has no zeros in the real line
- degree of $Q(z) > 1+$ degree of $P(z)$.

Then Cauchy's principle value integral can be evaluated using Cauchy's residue theorem.

Evaluation of certain contour integrals: type II

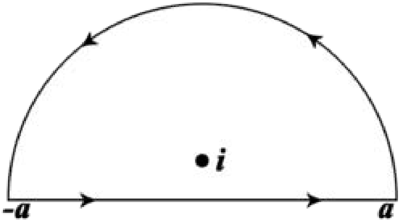
Type II Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx,$$

To evaluate this integral, we consider the complex-valued function

$$f(z) = \frac{1}{(z^2+1)^2},$$

which has singularities at i and $-i$. Consider the contour C like semicircle as shown below.



Evaluation of certain contour integrals: type II

Note that:

$$\int_C f(z)\,dz = \int_{-a}^{a} f(z)\,dz + \int_{\text{Arc}} f(z)\,dz$$

$$\int_{-a}^{a} f(z)\,dz = \int_C f(z)\,dz - \int_{\text{Arc}} f(z)\,dz$$

Furthermore, observe that

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}.$$

Then, by using Residue Theorem,

$$\int_C f(z)\,dz = \int_C \frac{\frac{1}{(z+i)^2}}{(z-i)^2}\,dz = 2\pi i \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\Bigg|_{z=i} = \frac{\pi}{2}$$

Evaluation of certain contour integrals: type II

If we call the arc of the semicircle 'Arc', we need to show that the integral over 'Arc' tends to zero as $a \to \infty$ using ML inequality

$$\left| \int_{\text{Arc}} f(z)\, dz \right| \leq ML,$$

where M is an upper bound of $|f(z)|$ along the Arc and L the length of Arc. Now, we have

$$\left| \int_{\text{Arc}} f(z)\, dz \right| \leq \frac{a\pi}{(a^2-1)^2} \to 0 \text{ as } a \to \infty.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx = \int_{-\infty}^{\infty} f(z)\, dz = \lim_{a \to +\infty} \int_{-a}^{a} f(z)\, dz = \frac{\pi}{2}.$$

Evaluation of certain contour integrals: type III

Type III Integrals of the form

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx \; dx \text{ or P. V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx \; dx,$$

where

- $P(x), Q(x)$ are real polynomials and $m > 0$,
- $Q(x)$ has no zeros in the real line,
- degree of $Q(x) >$ degree of $P(x)$.

Then

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx \; dx \text{ or P. V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx \; dx$$

can be evaluated using Cauchy's residue theorem.

Evaluation of certain contour integrals: type III

- **Jordan's Lemma:** If $0 < \theta \le \frac{\pi}{2}$, then $\frac{2\theta}{\pi} \le \sin\theta \le \theta$.

  Proof: Define $\phi(\theta) = \frac{\sin\theta}{\theta}$. Then $\phi'(\theta) = \frac{\psi(\theta)}{\theta^2}$, where $\psi(\theta) = \theta\cos\theta - \sin\theta$.

  1. Since $\psi(0) = 0$ and $\psi'(\theta) = -\theta\sin\theta \le 0$ for $0 < \theta \le \frac{\pi}{2}$, ψ decreases as θ increases i.e. $\psi(\theta) \le \psi(0) = 0$ for $0 < \theta \le \frac{\pi}{2}$.
  2. So $\phi'(\theta) = \frac{\psi(\theta)}{\theta^2} \le 0$ for $0 < \theta \le \frac{\pi}{2}$.
  3. That means ϕ is decreasing and hence $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$ for $0 < \theta \le \frac{\pi}{2}$.

- By Jordan's, lemma we have

$$\begin{aligned}\int_0^{\pi} e^{-a\sin\theta}d\theta &= \int_0^{\frac{\pi}{2}} e^{-a\sin\theta}d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-a\sin\theta}d\theta \\ &\le \int_0^{\frac{\pi}{2}} e^{-a\frac{2\theta}{\pi}}d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-a\frac{2(\pi-\theta)}{\pi}}d\theta.\end{aligned}$$

Here both the integrals in the RHS goes to 0 as $a \to \infty$.

Evaluation of certain contour integrals: type III

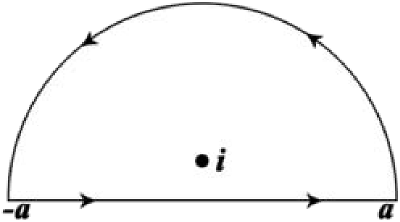
Evaluate:

$$\int_{-\infty}^{\infty} \frac{\cos tx}{x^2+1}\,dx \text{ or } \int_{-\infty}^{\infty} \frac{\sin tx}{x^2+1}\,dx$$

Consider the integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx$$

We will evaluate it by expressing it as a limit of contour integrals along the contour C. This contour starts along the real line from $-a$ to a, and then goes counterclockwise along a semicircle centered at 0 from a to $-a$. Let's assume that $a > 1$ so that the point i is enclosed by the curve.



Evaluation of certain contour integrals: type III

$$\operatorname{Res}\left(\frac{e^{itz}}{z^2+1}, i\right) = \lim_{z\to i}(z-i)\frac{e^{itz}}{z^2+1} = \lim_{z\to i}\frac{e^{itz}}{z+i} = \frac{e^{-t}}{2i}.$$

So by residue theorem

$$\int_C f(z)\,dz = (2\pi i)\operatorname{Res}_{z=i} f(z) = 2\pi i\frac{e^{-t}}{2i} = \pi e^{-t}.$$

The contour C may be split into a "straight" part and a curved arc, so that

$$\int_{\text{straight}} + \int_{\text{arc}} = \pi e^{-t}.$$

Thus,

$$\int_{-a}^{a}\frac{e^{itx}}{x^2+1}\,dx = \pi e^{-t} - \int_{\text{arc}}\frac{e^{itz}}{z^2+1}\,dz.$$

Evaluation of certain contour integrals: type III

$$
\begin{aligned}
\left|\int_{\text{arc}} \frac{e^{itz}}{z^2+1}\, dz\right| &\leq \int_0^{\pi} \left| a\frac{e^{ita(cos\theta + i\sin\theta)}}{a^2-1}\right| \, d\theta \\
&\leq \frac{a}{a^2-1}\int_0^{\pi} e^{-ta\sin\theta}\, d\theta.
\end{aligned}
$$

Hence,

$$
\left|\int_{\text{arc}} \frac{e^{itz}}{z^2+1}\, dz\right| \to 0 \text{ as } a \to \infty
$$

and

$$
\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos tx}{x^2+1}\, dx &= \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{itx}}{x^2+1}\, dx \\
&= \lim_{a\to\infty} \left[\pi e^{-t} - \int_{\text{arc}} \frac{e^{itz}}{z^2+1}\, dz\right] \\
&= \pi e^{-t}.
\end{aligned}
$$