Evaluation of integrals - II

Evaluation of certain contour integrals: Type IV

Type IV Integrals of the form

$$\int_0^\infty \frac{\sin x}{x} dx$$

can be evaluated using Cauchy's residue theorem. In order to handle the improper integral mentioned above, we require the following result.

Lemma: Suppose f has a simple pole at $z = a$ on the real axis. If c_ρ is the contour defined by $c_\rho(t) = a + \rho e^{i(\pi - t)},\ t \in (0, \pi)$, then

$$\lim_{\rho \to 0} \int_{c_\rho} f(z) dz = -i\pi \text{Res}(f, a).$$

Proof: Since f has a simple pole at $z = a$, the Laurent series expansion of f about $z = a$ is of the form

$$f(z) = \frac{\text{Res}(f, a)}{z - a} + g(z).$$

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Then,

$$\begin{aligned}\int_{c_\rho} f(z)dz &= \int_{c_\rho} \frac{\text{Res}(f,a)}{z-a}dz + \int_{c_\rho} g(z)dz \\ &= -\text{Res}(f,a)\int_0^\pi \frac{i\rho e^{i(\pi-t)}}{\rho e^{i(\pi-t)}}dt - \int_0^\pi g(a+\rho e^{i(\pi-t)})i\rho e^{i(\pi-t)}dt \\ &= -i\pi\text{Res}(f,a) - \int_0^\pi g(a+\rho e^{i(\pi-t)})i\rho e^{i(\pi-t)}dt.\end{aligned}$$

Note that f has Laurent series expansion in $0 < |z-a| < R$ for some $R > 0$. Then g must be continuous on $|z-a| \leq \rho_0$ for every $\rho < \rho_0 < R$, and $|g(z)| < M$ on $|z-a| \leq \rho_0$ for some $M > 0$. Hence,

$$\left|\int_0^\pi g(a+\rho e^{i(\pi-t)})i\rho e^{i(\pi-t)}dt\right| \leq \rho M\pi \to 0 \text{ as } \rho \to 0.$$

Thus,

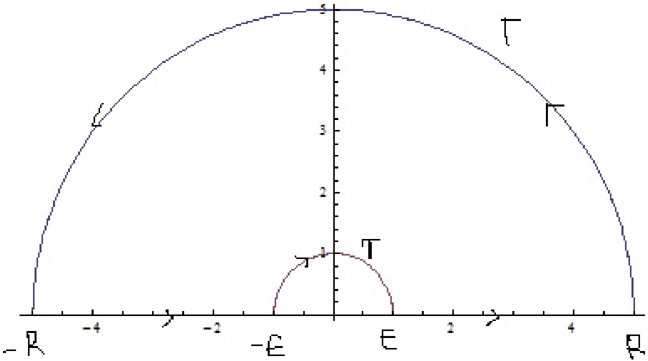
$$\lim_{\rho\to 0}\int_{c_\rho} f(z)dz = -i\pi\text{Res}(f,a).$$

Evaluation of certain contour integrals: Type IV

Consider the improper integral

$$\int_0^\infty \frac{\sin x}{x} dx.$$

Define $f(z) = \frac{e^{iz}}{z}$, ($z = 0$ is a simple pole on the real axis).
Consider the contour $C = [-R, -\epsilon] \cup \tau \cup [\epsilon, R] \cup \Gamma$.



Evaluation of certain contour integrals: Type IV

By Cauchy's theorem

$$\int_C \frac{e^{iz}}{z}dz = \int_{[-R,-\epsilon]} \frac{e^{iz}}{z}dz + \int_\tau \frac{e^{iz}}{z}dz + \int_{[\epsilon,R]} \frac{e^{iz}}{z}dz + \int_\Gamma \frac{e^{iz}}{z}dz = 0.$$

But

$$\int_{[-R,-\epsilon]} \frac{e^{iz}}{z}dz + \int_{[\epsilon,R]} \frac{e^{iz}}{z}dz = \int_{[\epsilon,R]} \frac{e^{ix} - e^{-ix}}{x}dx$$

So

$$\int_{[\epsilon,R]} \frac{e^{ix} - e^{-ix}}{x}dx = -\int_\tau \frac{e^{iz}}{z}dz - \int_\Gamma \frac{e^{iz}}{z}dz \to i\pi$$

as $\epsilon \to 0$ (by the previous Lemma) and $R \to \infty$ (by Jordan's inequality) and hence,

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

Evaluation of certain contour integrals: Type V

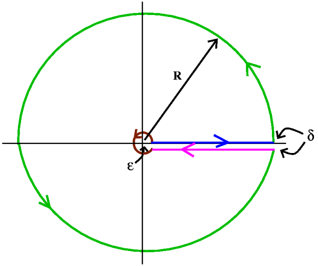
Integration along a branch cut: Consider the improper integral

$$\int_0^\infty \frac{x^{-a}}{1+x}dx, \text{ where } 0 < a < 1.$$

Define

$$f(z) = \frac{z^{-a}}{1+z}, \text{ where } |z| > 0 \text{ and } 0 < \arg z < 2\pi.$$

- The function $\frac{z^{-a}}{1+z}$ is a multi-valued function with a branch cut along $\arg z = 0$ (the positive real axis).
- Consider the contour $C = [\epsilon + i\delta, R + i\delta] \cup \Gamma_R \cup [R - i\delta, \epsilon - i\delta] \cup \{-\gamma_\epsilon\}$.



Evaluation of certain contour integrals: Type V

By residue theorem, we get

$$\left(\int_{[\epsilon+i\delta, R+i\delta]} + \int_{\Gamma_R} + \int_{[R-i\delta, \epsilon-i\delta]} + \int_{-\gamma_\epsilon} \right) f(z)dz = 2\pi i \text{Res}(f, -1) = 2\pi i e^{-ia\pi}.$$

Since

$$f(z) = \frac{exp(-a \log z)}{z+1} = \frac{exp(-a(\ln r + i\theta))}{re^{i\theta}+1},$$

where $z = re^{i\theta}$, it follows that,
on the segment $[\epsilon + i\delta, R + i\delta]$, $\theta \to 0$ as $\delta \to 0$,

$$f(z) \to \frac{exp(-a(\ln r + i.0))}{re^{i.0}+1} = \frac{r^{-a}}{1+r} \quad \text{as} \quad \delta \to 0,$$

whereas, on $[R - i\delta, \epsilon - i\delta]$, $\theta \to 2\pi$ as $\delta \to 0$,

$$f(z) \to \frac{exp(-a(\ln r + i.2\pi))}{re^{i.2\pi}+1} = \frac{r^{-a}}{1+r} e^{-2a\pi i} \quad \text{as} \quad \delta \to 0.$$

Evaluation of certain contour integrals: Type V

On the other hand, we get

$$\left|\int_{\Gamma_R} \frac{z^{-a}}{1+z}dz\right| \leq \frac{R^{-a}}{R-1}2\pi R = \frac{2\pi R}{R-1}\frac{1}{R^a} \to 0 \text{ as } R \to \infty$$

and

$$\left|\int_{\gamma_\epsilon} \frac{z^{-a}}{1+z}dz\right| \leq \frac{\epsilon^{-a}}{\epsilon-1}2\pi\epsilon = \frac{2\pi}{1-\epsilon}\epsilon^{1-a} \to 0 \text{ as } \epsilon \to 0.$$

Hence,

$$\lim_{R\to\infty,\epsilon\to 0}\left(\int_\epsilon^R \frac{r^{-a}}{1+r}dr + \int_R^\epsilon \frac{r^{-a}}{1+r}e^{-2a\pi i}dr\right) = 2\pi i e^{-ia\pi}$$

That is,

$$(1-e^{-2a\pi i})\int_0^\infty \frac{r^{-a}}{1+r}dr = 2\pi i e^{-ia\pi},$$

and hence

$$\int_0^\infty \frac{r^{-a}}{1+r}dr = \frac{2\pi i e^{-ia\pi}}{(1-e^{-2a\pi i})} = \frac{\pi}{\sin a\pi} \quad (0 < a < 1).$$

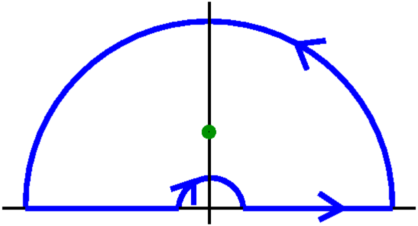
Integration around a branch cut (continue):
Consider the improper integral

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx.$$

Define

$$f(z) = \frac{\log z}{1+z^2}, \text{ where } |z| > 0 \text{ and } -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}.$$

- The function $\frac{\log z}{1+z^2}$ is a multi-valued function whose branch cut consists of origin and negative imaginary axis.
- Consider the contour $C = [\epsilon, R] \cup \Gamma_R \cup [-R, -\epsilon] \cup \{-\gamma_\epsilon\}$.



Evaluation of certain contour integrals: Type VI

By Cauchy's residue theorem

$$\left(\int_{[\epsilon,R]}+\int_{\Gamma_R}+\int_{[-R,-\epsilon]}+\int_{-\gamma_\epsilon}\right) f(z)dz = 2\pi i \mathrm{Res}(f,i) = 2\pi i\frac{\pi}{4} = \frac{\pi^2 i}{2}.$$

Since

$$f(z) = \frac{\log z}{z^2+1} = \frac{\log|z| + i\theta}{r^2 e^{2i\theta}+1},$$

where $z = re^{i\theta}$, it follows that
on the line segment $[\epsilon, R]$, $\theta = 0$,

$$f(z) = \frac{\log x}{x^2+1},$$

whereas, on $[-R,-\epsilon]$, $\theta = \pi$,

$$f(z) = \frac{\log|x| + i\pi}{x^2+1}.$$

Evaluation of certain contour integrals: Type VI

But

$$\left|\int_{\Gamma_R} \frac{\log z}{1+z^2}dz\right| = \left|\int_{\Gamma_R} \frac{\log R + i\theta}{1+R^2e^{2i\theta}} iRe^{i\theta}d\theta\right|$$
$$\leq R\frac{|\log R|}{R^2-1}\pi + \frac{R}{R^2-1}\int_0^{\pi} \theta d\theta \to 0$$

as $R \to \infty$, and

$$\left|\int_{\gamma_\epsilon} \frac{\log z}{1+z^2}dz\right| = \left|\int_{\gamma_\epsilon} \frac{\log \epsilon + i\theta}{1+\epsilon^2e^{2i\theta}} i\epsilon e^{i\theta}d\theta\right|$$
$$\leq \epsilon\pi\frac{|\log \epsilon|}{\epsilon^2-1} + \frac{\epsilon}{\epsilon^2-1}\int_0^{\pi} \theta d\theta \to 0 \text{ as } \epsilon \to 0.$$

Evaluation of certain contour integrals: Type VI

Hence,

$$\lim_{R\to\infty,\epsilon\to 0}\left(\int_{\epsilon}^{R}\frac{\log x}{x^2+1}dx+\int_{-R}^{-\epsilon}\frac{\log|x|+i\pi}{x^2+1}dx\right)=\frac{\pi^2 i}{2}$$

That is,

$$\lim_{R\to\infty,\epsilon\to 0}\left(\int_{\epsilon}^{R}\frac{\log x}{x^2+1}dx+\int_{\epsilon}^{R}\frac{\log|x|}{x^2+1}dx+\int_{\epsilon}^{R}\frac{i\pi}{x^2+1}dx\right)=\frac{\pi^2 i}{2}.$$

Thus,

$$\int_0^{\infty}\frac{\log x}{x^2+1}dx=0$$

and

$$\int_0^{\infty}\frac{1}{x^2+1}dx=\frac{\pi}{2}.$$