Conformal Mapping

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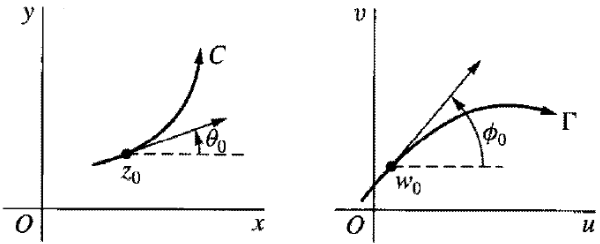
Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve in the domain D. Suppose that $f(z)$ is a function defined for all points z on γ. If we let C denote the image of γ under the transformation $w = f(z)$, then the parametric equation of C is given by $C(t) = w(t) = f(\gamma(t)),\ t \in [a, b]$.

- Suppose that γ passes through $z_0 = \gamma(t_0),\ (a < t_0 < b)$ at which f is analytic and $f'(z_0) \neq 0$. Then

$$w'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0).$$

- That means

$$\arg C'(t_0) = \arg\ w'(t_0) = \arg\ f'(z_0) + \arg\ \gamma'(t_0).$$



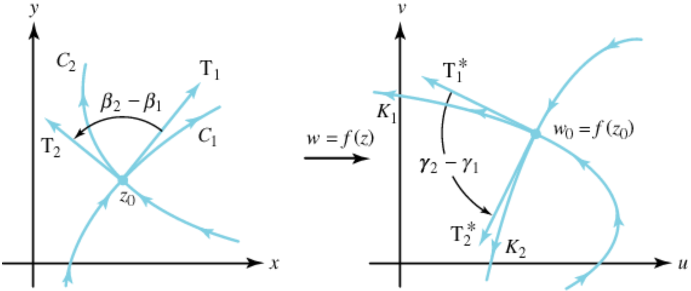
Conformal Mapping

Let $C_1, C_2 : [a, b] \to \mathbb{C}$ be two smooth curves in a domain D passing through z_0. Then by the above we have

$$\arg\ w_1'(t_0) = \arg\ f'(z_0) + \arg\ C_1'(t_0) \text{ and } \arg\ w_2'(t_0) = \arg\ f'(z_0) + \arg\ C_2'(t_0)$$

that means

$$\arg\ w_2'(t_0) - \arg\ w_1'(t_0) = \arg\ C_2'(t_0) - \arg\ C_1'(t_0).$$



Conformal Mapping

Definition: A transformation $w = f(z)$ is said to be conformal if it preserves the angle between oriented curves in magnitude as well as in orientation.

Note: From the above observation, if f is analytic in a domain D and $z_0 \in D$ with $f'(z_0) \neq 0$, then f is conformal at z_0.

- The function $f(z) = \bar{z}$ does not qualify as a conformal map since it only preserves the magnitude of the angle between two smooth curves, not their **orientation**. Such transformations are referred to as **isogonal mappings**.
- Let $f(z) = \bar{z}$. Then f is not a conformal map as it preserves only the magnitude of the angle between the two smooth curves but not **orientation**. Such transformations are called **isogonal mapping**.
- Let $f(z) = e^z$. Then f is a conformal at every point in $\mathbb{C}$ as $f'(z) = f(z) = e^z \neq 0$ for each $z \in \mathbb{C}$.
- Let $f(z) = \sin z$. Then f is a conformal map on $\mathbb{C} \setminus \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$.

Scale Factors

Definition: If f is analytic at z_0 and $f'(z_0) = 0$, then the point z_0 is called a critical point of f.

The following theorem gives the behavior of an analytic function in a neighborhood of critical point:

Theorem: Let f be analytic at z_0. If $f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$ and $f^k(z_0) \neq 0$, then the mapping $w = f(z)$ magnifies the angle at the vertex z_0 by a factor k.

Proof. Since f is analytic at z_0, we have

$$\begin{aligned} f(z) &= f(z_0) + (z - z_0)f'(z_0) + \frac{1}{2!}(z - z_0)^2 f''(z_0) + \cdots \\ &= f(z_0) + (z - z_0)^k \left[\frac{1}{k!} f^k(z_0) + \frac{1}{(k+1)!}(z - z_0) f^{k+1}(z_0) + \cdots \right] \\ &= f(z_0) + (z - z_0)^k g(z) \text{ (say).} \end{aligned}$$

That means

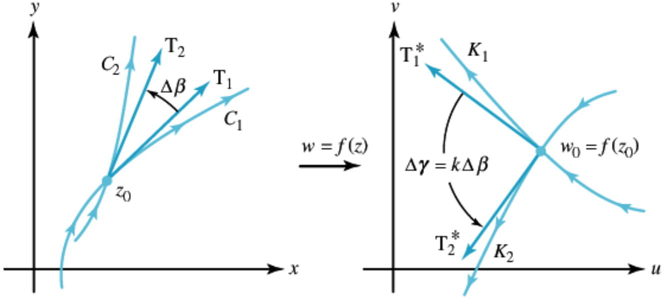
$$f(z) - f(z_0) = (z - z_0)^k g(z)$$

and

$$\arg\,(w - w_0) = \arg\,(f(z) - f(z_0)) = k \arg\,(z - z_0) + \arg\, g(z).$$

Scale Factors

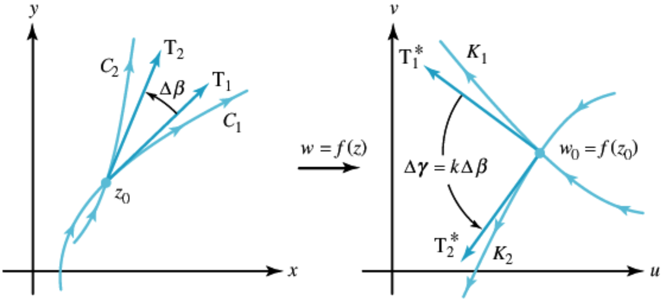
- Let C_1 and C_2 be two smooth curves passing through z_0.
- Let the image curves be $K_1 = f(C_1)$ and $K_2 = f(C_2)$.
- If z_1 is a variable point approaching to z_0 along C_1, then $w_1 = f(z_1)$ will approach to $w_0 = f(z_0)$ along the image curves K_1.
- Similarly, if z_2 is a variable point approaching to z_0 along C_2, then $w_2 = f(z_2)$ will approach to $w_0 = f(z_0)$ along the image curves K_2
- Let θ and Φ
  be the angle between C_1, C_2 at z_0 and between K_1, K_2 at $f(z_0)$ respectively.



Scale Factors

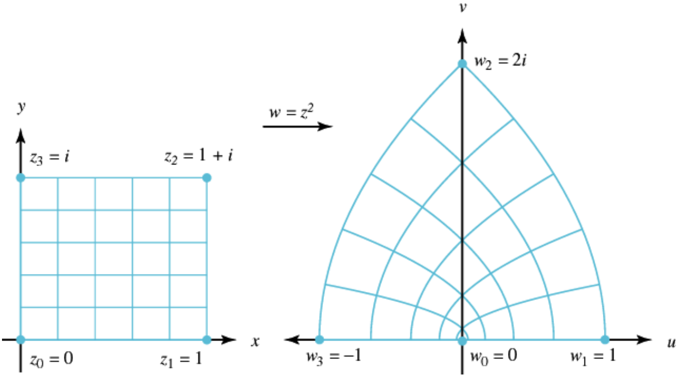
Then

$$
\begin{aligned}
\Phi &= \lim_{z_1, z_2 \to z_0} \arg \left(\frac{f(z_1) - f(z_0)}{f(z_2) - f(z_0)} \right) \\
&= \lim_{z_1, z_2 \to z_0} \arg \left(\frac{(z_1 - z_0)^k g(z_1)}{(z_2 - z_0)^k g(z_2)} \right) \\
&= \lim_{z_1, z_2 \to z_0} \left[k \arg \frac{z_1 - z_0}{z_2 - z_0} + \arg \frac{g(z_1)}{g(z_2)} \right] \\
&= k\theta.
\end{aligned}
$$



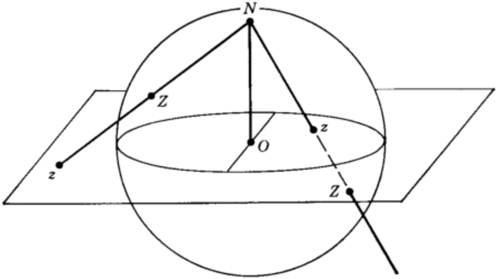
Question: Find the image of the unit square $S = \{x + iy : 0 < x < 1, 0 < y < 1\}$ under the map $w = f(z) = z^2$?

Answer: The map $f(z) = z^2$ is conformal at all $z \neq 0$. So the vertices $z_1 = 1, z_2 = 1 + i$ and $z_3 = i$ are mapped onto right angles at the vertices $w_1 = 1, w_2 = 2i$ and $w_3 = i$. But $f''(0) = 2 \neq 0$. So, the angle at the vertex z_0 is magnified by factor 2.



The extended complex plane and Riemann Sphere

- Let S^2 denote the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in R^3 and let $N = (0, 0, 1)$ denotes the "north pole" of S^2. Identify $\mathbb{C}$ with $\{(x_1, x_2, 0) : (x_1, x_2) \in \mathbb{R}^2\}$ so that $\mathbb{C}$ cuts S^2 in the equator.
- For each $z \in \mathbb{C}$, consider the straight line in $\mathbb{R}^3$ through z and N. This straight line intersects the sphere at precisely one point $Z \neq N$.
- **Question:** What happens to Z as $|z| \to \infty$?
  Answer: The point Z approaches N.
- Hence, we identify N with the point ∞ in $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Thus, the extended complex plane $\mathbb{C}_\infty$ can be represented as the sphere S^2, which is known as the **Riemann Sphere**.



The extended plane and its spherical representation

- Let $z = x + iy \in \mathbb{C}$, and $Z = (X_1, X_2, X_3)$ be the corresponding point on S^2.
- We want to find X_1, X_2 and X_3 in terms of x and y.
- The parametric equation of the straight line in $\mathbb{R}^3$ through z and N is given by

$$\{tN + (1-t)z : t \in \mathbb{R}\} \text{ i.e. } \{((1-t)x, (1-t)y, t) : t \in \mathbb{R}\}.$$

- Since this line intersects S^2 at $Z = ((1-t_0)x, (1-t_0)y, t_0)$, we have

$$1 = (1-t_0)^2 x^2 + (1-t_0)^2 y^2 + t_0^2 \Longrightarrow t_0 = \frac{x^2+y^2-1}{x^2+y^2+1}.$$

  Thus,

$$X_1 = \frac{2x}{x^2+y^2+1}, \; X_2 = \frac{2y}{x^2+y^2+1} \text{ and } X_3 = \frac{x^2+y^2-1}{x^2+y^2+1}.$$

The extended plane and its spherical representation

- Now, we will write x and y in terms of X_1, X_2 and X_3.
- From the last observation that

$$1 - X_3 = \frac{2}{x^2 + y^2 + 1}.$$

- Given the point $Z \in S^2$, the corresponding point $z = x + iy \in \mathbb{C}$ is

$$x = \frac{X_1}{1 - X_3} \quad \text{and} \quad y = \frac{X_2}{1 - X_3}.$$

- The correspondence between S^2 and $\mathbb{C}_\infty$ is defined by

$$Z = (X_1, X_2, X_3) \mapsto z = (x + iy)$$

is called **stereographic projection**.

Möbius transformations

If a, b, c and d are complex constants such that $ad - bc \neq 0$, then the function

$$w = S(z) = \frac{az + b}{cz + d}$$

is called a Möbius transformation. It is also known as a bilinear transformation or a linear fractional transformation.

Observation:

- Every Möbius transformation is a conformal map.

$$S'(z) = \frac{(cz + d)a - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} \neq 0.$$

- The map S is analytic on $\mathbb{C} \setminus \{-\frac{d}{c}\}$.
- Composition of two Möbius transformation is a Möbius transformation.

Möbius transformations

- Define $T(w) = \dfrac{-dw+b}{cw-a}$ then $SoT(w) = w$ and $ToS(z) = z$. So S is invertible and $S^{-1} = T$.
- The map $S : \mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C} \setminus \{-\frac{a}{c}\}$ is one one and onto. If we define $S(-\frac{d}{c}) = \infty$ and $S(\infty) = \dfrac{a}{c}$, then

$$S : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

is a bijection.

Types of Möbius transformations:

- Let $a \in \mathbb{C}$. Then $S(z) = z + a$ is called a *translation* map.
- Let $0 \neq a \in \mathbb{C}$. Then $S(z) = az$ is called a *dilation* map. (If $|a| < 1$, S is called a contraction map, and if $|a| > 1$, S is a called expansion map.)
- Let $\theta \in \mathbb{R}$. Then $S(z) = e^{i\theta} z$ is called a *rotation* map.
- The map $S(z) = \frac{1}{z}$ is called an *inversion* map.