Möbius transformation and its applications

- Every Möbius transformation is the composition of the translation, dilation and inversion.

  Proof. Let $w = S(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ be a Möbius transformation.

  First, suppose $c = 0$. Hence $S(z) = (a/d)z + (b/d)$. If

$$S_1(z) = (a/d)z, S_2(z) = z + (b/d),$$

  then $S_2 \circ S_1 = S$, and we are done. Now, let $c \neq 0$, then

$$S_1(z) = z + d/c, S_2(z) = 1/z, S_3(z) = \frac{bc - ad}{c^2} z, \; S_4(z) = z + a/c.$$

  Then

$$S_4 \circ S_3 \circ S_2 \circ S_1(z) = S(z) = \frac{az+b}{cz+d}.$$

Möbius transformations

- A point $z_0 \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is called a fixed point of a function f if $f(z_0) = z_0$.
- **Question:** What are the fixed points of a Möbius transformation S?
  Answer:
- If z satisfies the condition

$$S(z) = \frac{az+b}{cz+d} = z,$$

then

$$cz^2 - (a-d)z - b = 0.$$

- A Möbius transformation can have at most two fixed points unless it is the identity map.
- If $c = 0$, then $z = -\frac{b}{a-d}$ (∞ if $a = d$) is the only fixed point of S .

Möbius transformations

Question: How many Möbius transformations are possible by its action on three distinct points in $\mathbb{C}_\infty$?

Answer: One!

Proof. Let S and T be two Möbius transformations such that

$$S(a) = T(a) = \alpha, \quad S(b) = T(b) = \beta \text{ and } \quad S(c) = T(c) = \gamma,$$

where a, b, c are three distinct points in $\mathbb{C}_\infty$. Consider

$$T^{-1} \circ S(a) = a, T^{-1} \circ S(b) = b \text{ and } T^{-1} \circ S(c) = c.$$

So we have a Möbius transformation $T^{-1} \circ S$ having three fixed points. Hence, $T^{-1} \circ S = I$. That is $S = T$.

Question: How to find a Möbius transformation if its action on three distinct points in $\mathbb{C}_\infty$ is given?

Möbius transformations

Definition: Given four distinct points $z_j \in \mathbb{C}_\infty\, ; j = 1, .., 4$, the cross ratio of z_j's is defined by

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

- If $z_2 = \infty$, then $(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)}{(z_1 - z_4)}$
- If $z_3 = \infty$, then $(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_4)}{(z_1 - z_4)}$
- If $z_4 = \infty$, then $(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)}{(z_2 - z_3)}$
- The cross ratio defines a Möbius transformation via

$$S(z) = (z, z_2, z_3, z_4) = \frac{(z - z_3)(z_2 - z_4)}{(z_2 - z_3)(z - z_4)}$$

such that $S(z_2) = 1, S(z_3) = 0$ and $S(z_4) = \infty$.

Result: The cross ratio is invariant under Möbius transformation. i. e. if $z_j \in \mathbb{C}_\infty\,; j = 1, .., 4$ are four distinct points and T is any Möbius transformation, then

$$(z_1, z_2, z_3, z_4) = (T(z_1), T(z_2), T(z_3), T(z_4)).$$

Proof. We know that $S(z) = (z, z_2, z_3, z_4)$ is a Möbius transformation such that $S(z_2) = 1, S(z_3) = 0$ and $S(z_4) = \infty$. Then ST^{-1} is a Möbius transformation such that

$$ST^{-1}(Tz_2) = S(z_2) = 1, ST^{-1}(Tz_3) = S(z_3) = 0 \text{ and } ST^{-1}(Tz_4) = S(z_4) = \infty.$$

Since a Möbius transformation is uniquely determined by its action on three distinct points in $\mathbb{C}_\infty$, we have

$$ST^{-1}(z) = (z, T(z_2), T(z_3), T(z_4)).$$

So

$$ST^{-1}(T(z_1)) = S(z_1) = (z_1, z_2, z_3, z_4) = (T(z_1), T(z_2), T(z_3), T(z_4)).$$

Möbius transformations

Result: If z_2, z_3, z_4 are three distinct points in $\mathbb{C}_\infty$ and if w_2, w_3, w_4 are also three distinct points of $\mathbb{C}_\infty$, then there is one and only one Möbius transformation S such that $Sz_2 = w_2, Sz_3 = w_3, Sz_4 = w_4$.

Proof

- Let $T(z) = (z, z_2, z_3, z_4)$, $M(z) = (z, w_2, w_3, w_4)$ and put $S = M^{-1} \circ T$.
- Clearly S has desired property.
- If R is another Möbius transformation with $Rz_j = w_j$ for $j = 2, 3, 4$, then $R^{-1} \circ S$ has three fixed points (z_2, z_3 and z_4).
- Hence $R^{-1} \circ S = I$. That is, $R = S$.

Question: Find a Möbius transformation that maps $z_1 = 1, z_2 = 0, z_3 = -1$ onto the points $w_1 = i, w_2 = \infty, w_3 = 1$.

Answer: We know that

$$(z, z_1, z_2, z_2) = (T(z), T(z_1), T(z_2), T(z_2)).$$

That is,

$$(z, 1, 0, -1) = (T(z), i, \infty, 1),$$

which on solving gives

$$T(z) = \frac{(i+1)z + (i-1)}{2z}.$$

Möbius transformations

Theorem. A Möbius transformation maps circle onto circle.
Proof.

- Recall that every Möbius transformation is the composition of the translation, dilation and inversion.
- It is easy to show that translation, dilations maps circle onto circle.
- To prove this result, it is enough to show that inversion maps circle onto circle.
- Consider the mapping $w = S(z) = \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$.
- If $w = u + iv$ and $z = x + iy$, then

$$u = \frac{x}{x^2 + y^2} \text{ and } v = -\frac{y}{x^2 + y^2}.$$

- Similarly,

$$x = \frac{u}{u^2 + v^2} \text{ and } y = -\frac{v}{u^2 + v^2}.$$

- The general equation of a circle is

$$a(x^2 + y^2) + bx + cy + d = 0. \quad (1)$$

- Applying transformation $w = \frac{1}{z}$ (i.e. substituting $x = \frac{u}{u^2+v^2}$ and $y = -\frac{v}{u^2+v^2}$) we have

$$a\left(\left(\frac{u}{u^2+v^2}\right)^2 + \left(-\frac{v}{u^2+v^2}\right)^2\right) + b\left(\frac{u}{u^2+v^2}\right) + c\left(-\frac{v}{u^2+v^2}\right) + d = 0$$

- Which on simplification reduces to

$$d(u^2 + v^2) + bu - cv + a = o \quad (2)$$

- Find a Möbius transformation that takes UHP to RHP.
- Find the image of unit disc under the map $w = f(z) = \frac{z}{1-z}$.
  Ans: Re $w > -\frac{1}{2}$.
- Find the image of $D = \{z : |z+1| < 1\}$ under the map

$$w = f(z) = \frac{(1-i)z+2}{(1+i)z+2}.$$