Complex Integration

Complex Integration

Integral of a complex valued function of real variable:

- **Definition:** Let $f : [a, b] \to \mathbb{C}$ be a function. Then $f(t) = u(t) + iv(t)$ where $u, v : [a, b] \to \mathbb{R}$. Define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt.$$

  If $U' = u$ and $V' = v$ and $F(t) = U(t) + iV(t)$, then by fundamental theorem of calculus $\int_a^b f(t)dt = F(b) - F(a)$.

- For $\alpha \in \mathbb{R}$, $\int_a^b e^{i\alpha t}dt = \dfrac{e^{i\alpha b} - e^{i\alpha a}}{i\alpha}$.

- $\int_0^1 (1 + it)^2 \, dt = \int_0^1 (1 - t^2) \, dt + i \int_0^1 2t \, dt = \dfrac{2}{3} + i$.

- If $f : [a, b] \to \mathbb{C}$ piecewise continuous, then the integral $\int_a^b f(t)dt$ exists.

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- $\text{Re}\left(\int_a^b f(t)dt\right) = \int_a^b \text{Re}\,(f(t))dt.$
- $\text{Im}\left(\int_a^b f(t)dt\right) = \int_a^b \text{Im}\,(f(t))dt.$
- $\int_a^b [f(t) \pm g(t)]dt = \int_a^b f(t)dt \pm \int_a^b g(t)dt.$
- $\int_a^b \alpha f(t)dt = \alpha \int_a^b f(t)dt, \quad \alpha \in \mathbb{C}$ and $\int_a^b f(t)dt = -\int_b^a f(t)dt.$
- $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$
- $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

  Proof: Let $\int_a^b f(t)dt = Re^{i\theta}$, then $R = e^{-i\theta}\int_a^b f(t)dt = \int_a^b e^{-i\theta} f(t)dt =$ $\text{Re}\left(\int_a^b e^{-i\theta} f(t)dt\right) = \int_a^b \text{Re}\,(e^{-i\theta} f(t))dt$. So

$$\left|\int_a^b f(t)dt\right| = R = \int_a^b \text{Re}\,(e^{-i\theta} f(t))dt \leq \int_a^b |e^{-i\theta} f(t)|dt \leq \int_a^b |f(t)|dt.$$

Complex integration

- **Definition:** A **curve** is a continuous function $\gamma : [a, b] \to \mathbb{C}$. So $\gamma(t) = x(t) + iy(t)$ with $x, y : [a, b] \to \mathbb{R}$.
- A curve γ is called a **smooth curve** if γ is differentiable and γ' is continuous and nonzero for all t.
- A **contour/piecewise smooth curve** is a smooth curve that is obtained by joining finitely many smooth curves end to end.
- $\gamma_1(t) = e^{it}, t \in [0, 1]; \quad \gamma_2(t) = (1 - t)a + tb, \; t \in [0, 1]$.
- **Definition:** A curve γ is **simple** if it does not intersect itself except possibly at end points. That means $\gamma(t_1) \neq \gamma(t_2)$ when $a < t_1 < t_2 < b$.
- **Definition:** A curve γ is said to be **closed** if $\gamma(a) = \gamma(b)$.
- **Definition:** A curve γ is simple and closed, then we say that γ is a **simple closed curve** or **Jordan curve**.

Complex integration

- Let γ be a piecewise smooth curve defined on $[a, b]$. The length of γ is given by

$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$

- **Orientation:** Let γ be a simple closed contour with parametrization $\gamma(t),\ t \in [a, b]$. As t moves from a to b, the curve γ moves in a specific direction called the orientation of the curve induced by the parametrization.
- **Convention:** If the interior of the bounded domain enclosed by γ is on the left as t moves from a to b, then we say the orientation is in the **positive sense** (counter clockwise or anticlockwise sense). Otherwise, γ is oriented **negatively** (clockwise direction).
- Let $\gamma : [a, b] \to \mathbb{C}$ be a curve , then the curve with the reverse orientation is denoted as $-\gamma$ and is defined as $-\gamma : [a, b] \to \mathbb{C}$, $-\gamma(t) = \gamma(b + a - t)$.
- $\gamma(t) = e^{it},\ t \in [0, 2\pi]$ (Positive orientation)where as $\gamma(t) = e^{i(2\pi - t)},\ t \in [0, 2\pi]$ (Negative orientation)

Complex integration

Definition: Let $\gamma(t)$; $t \in [a, b]$, be a contour and f be a complex valued continuous function defined on a set containing γ , then the **line integral or the contour integral** of f along the curve γ is defined by

$$\int_{\gamma} f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Example: Let $f(z) = \bar{z}$.

- If $\gamma_1(t) = e^{it}, \ t \in [0, \pi]$, then

$$\int_{\gamma_1} \bar{z}dz = \int_0^{\pi} \overline{\gamma_1(t)}\gamma_1'(t)dt = \int_0^{\pi} e^{-it}(i)e^{it}dt = i\pi.$$

- If $\gamma_2(t) = 1(1-t) + t.(-1) = 1 - 2t, \ t \in [0, 1]$, then

$$\int_{\gamma_2} \bar{z}dz = \int_0^{1} \overline{\gamma_2(t)}\gamma_2'(t)dt = \int_0^{1} [1-2t](-2)dt = 0.$$

- In the above example γ_1 and γ_2 are two paths joining 1 and -1. But the line integral along the paths γ_1 and γ_2 are NOT same.
- **Question:** When a line integral of f does not depend on path?

Complex integration

- (The fundamental integral) For $n \in \mathbb{Z}$

$$\int_{|z|=1} z^{n-1} dz = \begin{cases} 0 & \text{if} \quad n \neq 0 \\ 2\pi i & \text{if} \quad n = 0. \end{cases}$$

- Let f, g be piecewise continuous complex valued functions, then $\int_\gamma [\alpha f \pm g](z) dz = \alpha \int_\gamma f(z) dz \pm \int_\gamma g(z) dz.$
- Let $\gamma : [a, b] \to \mathbb{C}$ be a curve and $a < c < b$. If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$, then $\int_\gamma f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz.$
- $\int_{-\gamma} f(z) dz = -\int_\gamma f(z) dz.$

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- Let f be a piecewise continuous function and γ be a curve defining the contour γ. If $|f(z)| \leq M$ for all $z \in \gamma$ and $L =$length of γ , then

$$\left|\int_{\gamma} f(z)dz\right| \leq \int_{a}^{b} |f(\gamma(t))||\gamma'(t)|dt \leq M \int_{a}^{b} |\gamma'(t)|dt = ML. \quad \textbf{(ML-inequality)}$$

- Let $\gamma(t) = 2e^{it}, t \in [0, \frac{\pi}{2}]$ and $f(z) = \dfrac{z+4}{z^3-1}$. Then by ML-ineuqality

$$\left|\int_{\gamma} f(z)\, dz\right| \leq \frac{6\pi}{7}.$$

Antiderivatives

Definition: The **antiderivative or primitive** of a continuous function f in a domain D is a function F such that $F'(z) = f(z)$ for all $z \in D$.
The primitive of a function is **unique** up to an additive constant.
(Answer to the **Question:** When a line integral of f does not depend on path?)

- **Theorem:** Let f be a continuous function defined on a domain D and $f(z)$ has antiderivative $F(z)$ in D. Let $z_1, z_2 \in D$. Then for any contour C lying in D starting from z_1, and ending at z_2, the value of the integral $\displaystyle\int_C f(z)dz$ is **independent of the contour**.
  Proof. Suppose that C is given by a map $\gamma : [a, b] \to \mathbb{C}$. Then $\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t)$. Hence

$$\begin{aligned}\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b \frac{d}{dt}F(\gamma(t))dt \\ &= F(\gamma(a)) - F(\gamma(b)) = F(z_2) - F(z_1).\end{aligned}$$

- **Note:** In particular, we have $\int_C f(z)dz = 0$ if C is a closed contour.

Antiderivatives

Whenever such F exists, we write

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2} F'(z)dz = F(z_1) - F(z_2).$$

1. $\displaystyle\int_{z_1}^{z_2} z^2 dz = \frac{z_2^3 - z_1^3}{3}.$
2. $\displaystyle\int_{-i\pi}^{i\pi} \cos z dz = \sin(i\pi) - \sin(-i\pi) = 2\sin(i\pi).$
3. $\displaystyle\int_{-i}^{i} \frac{1}{z} dz = \text{Log }(i) - \text{Log }(-i) = \frac{i\pi}{2} - \frac{-i\pi}{2} = i\pi.$
4. The function $\frac{1}{z^n}$, $n > 1$ is continuous on $\mathbb{C}^*$. Thus, the integral of the above function on any contour joining nonzero complex numbers z_1, z_2 not passing through origin is given by
$$\int_{z_1}^{z_2} \frac{dz}{z^n} = -(n-1)\left(\frac{1}{z_2^{n-1}} - \frac{1}{z_1^{n-1}}\right).$$
In particular we have $\displaystyle\int_C \frac{dz}{z^n} = 0$ where C any closed curve not possing through origin.