Cauchy's Theorem

Complex integration

Question: Under what conditions on f can guarantee the existence of F such that $F' = f$?

- **Definition:** (Simply connected domain) A domain D is called **simply connected** if every simple closed contour (within it) encloses points of D only.
- A domain D is called **multiply connected** if it is **not** simply connected.
- The sets $\mathbb{C}$, $\mathbb{D}$ and $RHP = \{z : \text{Re } z > 0\}$ are simply connected domains (they have no holes).
- But the sets $\mathbb{C}^*$, $\mathbb{D} \setminus \{0\}$, and the annulus $A_{a,b} = \{z \in \mathbb{C} : a < |z| < b\}$ are not simply connected domains.

Cauchy's Theorem

Theorem: If a function f is analytic on a simply connected domain D and C is a simple closed contour lying entirely in D, then

$$\int_C f(z)dz = 0.$$

We will prove the theorem under an extra hypothesis that f' is a continuous function.

[**Recall** (Green's Theorem) *Let C be a simple closed curve with positive orientation. Let R be the bounded region enclosed by C. If P and Q are continuous with continuous partial derivatives P_x, P_y, Q_x and Q_y within the interior of R, then*

$$\oint_C [P(x,y)dx + Q(x,y)]dy = \iint_R [Q_x(x,y) - P_y(x,y)]dxdy.$$

Proof. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ and $\gamma(t) = x(t) + iy(t)$, $a \leq t \leq b$ be the curve C. Then

$$\begin{aligned}
\int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)]dt \\
&= \int_a^b (ux' - vy')dt + i \int_a^b (vx' + uy')dt \\
&= \oint_C (udx - vdy) + i \oint_C (vdx + udy) \\
&= \iint_R (-v_x - u_y)dxdy + i \iint_R (u_x - v_y)dxdy, \\
&\qquad \text{(by Green's theorem)} \\
&= 0 \qquad \text{(by CR equations).}
\end{aligned}$$

Cauchy's Theorem

Let $\gamma(t) = e^{it}$, $-\pi < t \leq \pi$, and C denotes the circle of radius one with center at zero.

1. It follows from Cauchy's theorem that $\int_C f(z)dz = 0$, if $f(z) = e^{z^n}$, $\cos z$, or $\sin z$.
2. $\displaystyle\int_C f(z)dz = 0$ if $f(z) = \frac{1}{z^2}$, or $cosec^2 z$ from the fundamental theorem as $\frac{d}{dz}(-\frac{1}{z}) = \frac{1}{z^2}$ and $\frac{d}{dz}(-\cot z) = cosec^2 z$. Note that here Cauchy's theorem cannot be applied as the integrands are not analytic at zero.
3. $\displaystyle\int_C \frac{e^{iz^2}}{z^2+4} dz = 0$ by Cauchy's theorem. Note that the integrand is not analytic at $z = \pm 2$ but that does not bother us as these points are not enclosed by C.
4. If $f(z) = (\text{Im } z)^2$, then $\displaystyle\int_C f(z)dz = 0$ (**check this**). As f is not analytic anywhere in $\mathbb{C}$, Cauchy's theorem cannot be applied to prove this.

Consequences of Cauchy's Theorem

Independence of path:

- Let D be a simply connected domain and $f : D \to \mathbb{C}$ be analytic. Let z_1, z_2 be two points in D. If γ_1 and γ_2 be two simple curves joining z_1 and z_2 such that the curves lie entirely in D, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

To see this, consider the curve

$$\gamma(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq 1/2; \\ \gamma_2(2(1-t)), & 1/2 \leq t \leq 1. \end{cases}$$

(We have just reversed the direction of γ_2 and joined it with γ_1). Then γ is a simple closed curve and by Cauchy's theorem

$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{-\gamma_2} f(z)dz = 0,$$

which implies $\int\limits_{\gamma_1} f(z)dz = -\int\limits_{-\gamma_2} f(z)dz$. As $-\int\limits_{-\gamma_2} f(z)dz = \int\limits_{\gamma_2} f(z)dz$ we get the result.

Consequences of Cauchy's Theorem

Existence of antiderivative:

- If f is an analytic function on a simply connected domain D, then there exists a function F, which is analytic on D such that $F' = f$.

  Proof. For $z_0, z \in D$, define

  $$F(z) = \int_{z_0}^{z} f(w)dw.$$

  The above integral is considered as a contour integral over any curve lying in D and joining z with z_0. By the previous result, the integral is independent of any path joining z_0 and z, and hence the function F is well defined.

  We will show that $F' = f$. If $z + h \in D$, then

  $$F(z+h) - F(z) = \int_{z_0}^{z+h} f(w)dw - \int_{z_0}^{z} f(w)dw = \int_{z}^{z+h} f(w)dw,$$

  where the curve joining z and $z + h$ can be considered as a straight line $l(t) = z + th$, $t \in [0, 1]$.

Consequences of Cauchy's Theorem

Thus we get

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| = \frac{1}{|h|}\left|\int_z^{z+h}(f(w)-f(z))dw\right|,$$

(here we have used the fact that $\int_l dw = h$). Since f is continuous at z, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z+h)-f(z)| < \epsilon$ if $|h| < \delta$. Thus, for $|h| < \delta$, we get from ML inequality that

$$\frac{1}{|h|}\left|\int_z^{z+h}(f(w)-f(z))dw\right| \leq \frac{\epsilon|h|}{|h|}.$$

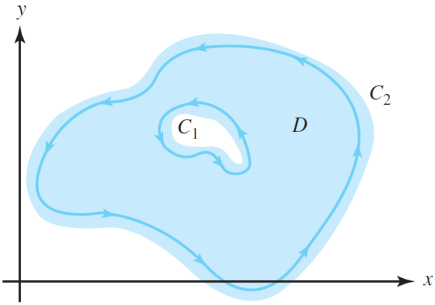
That is,

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z).$$

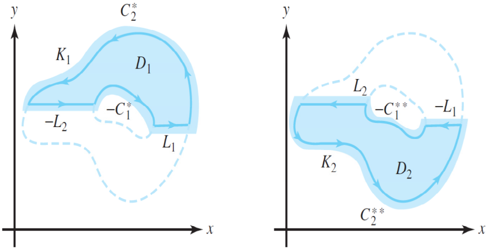
Deformation of contours

Theorem Let D be the domain bounded by two simple closed positively oriented contours C_1 and C_2 such that C_1 lies entirely in the interior region enclosed by C_2. If f is analytic in the domain D, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$



Proof. Assume that both C_1 and C_2 have positive (counterclockwise) orientation. We construct two disjoint contours or cuts L_1 and L_2 that join C_1 to C_2. The contour C_1 is cut into two contours C_1^* and C_1^{**} and the C_2 is cut into two contours C_2^* and C_2^{**}.



$$K_1 = -C_1^* + L_1 + C_2^* - L_2 \quad \text{and} \quad K_2 = -C_1^{**} + L_2 + C_2^{**} - L_1.$$

The function f is analytic on the simply connected domains D_1 and D_2 enclosed by simple closed curves K_1 and K_2, respectively.

Deformation of contours

By Cauchy's theorem,

$$\int_{K_1} f(z)dz = \int_{k_2} f(z)dz = 0.$$

Also $K_1 + K_2 = C_2^* + C_2^{**} - C_1^* - C_1^{**} = C_2 - C_1$. Thus

$$\int_{K_1+K_2} f(z)dz = \int_{C_2-C_1} f(z)dz = 0.$$

This implies that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Example: If C is any positively oriented simple closed contour surrounding (enclosing) the origin, then

$$\int_C \frac{1}{z}dz = 2\pi.i$$

Deformation of contours

Theorem Let $C, C_k;\ k = 1, 2, \ldots, n$ be simple closed positively oriented contours such that each C_k lies in the region enclosed by C and C_k has no common points with the interior enclosed by C_j if $k \neq j$. Let f be analytic on a domain D consisting of all points in the interior enclosed by C and exterior to each C_k. Then

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz.$$