Cauchy's Integral Formula

Cauchy's Integral Formula

Theorem Let f be analytic on a simply connected domain D. Suppose $z_0 \in D$ and C is a simple closed curve oriented counterclockwise lies entirely in D that encloses z_0. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \quad (\textbf{Cauchy's Integral Formula}).$$

Proof. Let $C(z_0, r)$ denotes the circle of radius r around z_0 for a sufficiently small $r > 0$ then

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz - f(z_0) \right| &= \left| \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \\ &= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}} ire^{i\theta} d\theta \right| \\ &\leq \frac{1}{2\pi} 2\pi \times \sup_{\theta \in [0, 2\pi]} |f(z_0 + re^{i\theta}) - f(z_0)| \\ &\quad (\text{ by } ML \text{ inequality}). \end{aligned}$$

As f is continuous, it follows that the righthand side goes to zero as r tends to zero.

- $\displaystyle\int_{|z-4|=5} \frac{\cos z}{z} dz = 2\pi i.$
- $\displaystyle\int_{|z-i|=1} \frac{z^2}{z^2+1} dz = -\pi.$
- Can we use Cauchy's integral formula to evaluate the following?

$$I = \int_{|z|=2} \frac{e^z}{z(z-1)} dz$$

Yes ! Write

$$I = \int_{C(0,2)} \frac{e^z}{z-1} dz - \int_{C(0,2)} \frac{e^z}{z} dz.$$

Now, apply Cauchy's integral formula, then we get the value of the integral equal to $2\pi i(e-1)$.

Cauchy's Integral Formula for higher derivatives

Theorem If f is analytic on a simply connected domain D, then f has derivatives of all orders in D (which are then analytic in D). For any $z_0 \in D$, one has

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

where C is a simple closed contour (oriented counterclockwise) around z_0 in D.

Proof: By Cauchy's integral formula

$$\begin{aligned}
f'(z_0) &= \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \\
&= \lim_{h \to 0} \frac{1}{2\pi i h} \int_C \left(\frac{f(z)}{z - z_0 - h} - \frac{f(z)}{z - z_0} \right) dz \\
&\quad (C \text{ is so chosen that the point } z_0 + h \text{ is enclosed by } C) \\
&= \lim_{h \to 0} \frac{1}{2\pi i h} \int_C \frac{f(z)h}{(z - z_0 - h)(z - z_0)} dz.
\end{aligned}$$

Cauchy's Integral Formula for higher derivatives

So we need to prove that

$$\left|\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)}\,dz - \int_C \frac{f(z)}{(z-z_0)^2}\,dz\right|$$
$$= \left|\int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2}\,dz\right| \to 0, \text{ as } \quad h \to 0.$$

We will use ML inequality to prove this. Now

- Let $|f(z)| \leq M$ for all $z \in C$.
- Let $\alpha = \min\{|z-z_0| : z \in C\}$, then $|z-z_0|^2 \geq \alpha^2$.
- $\alpha \leq |z-z_0| = |z-z_0-h+h| \leq |z-z_0-h| + |h|$.
- Hence for $|h| \leq \frac{\alpha}{2}$ we have $|z-z_0-h| \geq \alpha - |h| \geq \frac{\alpha}{2}$.

Cauchy's Integral Formula for higher derivatives

Therefore

$$\Big|\int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2}dz\Big| \leq \frac{M|h|l}{\frac{\alpha}{2}\alpha^2} = \frac{2M|h|l}{\alpha^3} \to 0,$$

as $h \to 0$.

By repeating exactly the same technique, we get

$$f^2(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}dz$$

and so on.

Cauchy's Integral Formula for higher derivatives

- $\displaystyle\int_{|z|=1} e^z z^{-3} dz = i\pi.$
- $\displaystyle\int_{|z-1|=5/2} \frac{1}{(z-4)(z+1)^4} dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3} \left(\frac{1}{z-4} \right) \Big|_{z=-1}.$

Summary Let C be a simple closed curve contained in a simply connected domain D, and f is an analytic function defined on D. Then

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \begin{cases} 2\pi i f(z_0), & \text{if } n = 0 \text{ and } z_0 \text{ is enclosed by } C. \\ \frac{2\pi i}{n!} f^n(z_0), & \text{if } n \geq 1 \text{ and } z_0 \text{ is enclosed by } C. \\ 0, & z_0 \text{ lies outside the region enclosed by } C. \end{cases}$$

Cauchy's estimate

Cauchy's estimate: Suppose f is analytic on a simply connected domain D and $\overline{B(z_0, R)} \subset D$ for some $R > 0$. If $|f(z)| \leq M$ for all $z \in B(z_0, R)$, then for all $n \geq 0$,

$$|f^n(z_0)| \leq \frac{n!M}{R^n},$$

where $\overline{B(z_0, R)} = \{z : |z - z_0| \leq R\}$.

Proof: From Cauchy's integral formula and ML inequality we have

$$\begin{aligned} |f^n(z_0)| &= \left|\frac{n!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z - z_0)^{n+1}} dz\right| \\ &\leq \frac{n!}{2\pi} M \frac{1}{R^{n+1}} 2\pi R = \frac{n!M}{R^n}. \end{aligned}$$

Liouville's Theorem

Liouville's Theorem: If f is analytic and bounded on the whole complex plane $\mathbb{C}$, then f is a constant function.

Proof: By Cauchy's estimate for any $z_0 \in \mathbb{C}$, we have

$$|f'(z_0)| \leq \frac{M}{R}$$

for all $R > 0$. This implies that $f'(z_0) = 0$. Since z_0 is arbitrary and hence $f' \equiv 0$. Therefore f is a constant function.

- $\sin z, \cos z, e^z$ etc., can not be bounded. If so, then by Liouville's theorem, they are constant.

Liouville's Theorem

- Does there exist a non-constant entire function f such that $e^{f(z)}$ is bounded?
- Does there exist a non-constant entire function f such that $\text{Re}(f)$ is bounded?
- Does there exist a non-constant entire function f such that $\text{Im}(f)$ is bounded?
- Does there exist a non-constant entire function f such that $f(x)$ is bounded for all real x?
- Does there exist a non-constant entire function f such that $|f(z)| > 1$ for all $z \in \mathbb{C}$?