MA15010H: Multi-variable Calculus

(Lecturenote 2: Sequential continuity and vector differentiability)

July - November, 2025

Sequential criterion of closed set: A set $S \subseteq \mathbb{R}^m$ is closed in \mathbb{R}^m if and only if $x \in S$ for every $x \in \mathbb{R}^m$ and for every sequence (x_n) in S with $x_n \to x$.

Proof: Let S be closed in \mathbb{R}^m and let (x_n) be a sequence in S such that $x_n \to x \in \mathbb{R}^m$. If possible, let $x \notin S$. Then $x \in \mathbb{R}^m \setminus S$ and since $\mathbb{R}^m \setminus S$ is open in \mathbb{R}^m , there exists r > 0 such that $B_r(x) \subseteq \mathbb{R}^m \setminus S$. Now, since $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_r(x)$ for all $n \geq n_0$. In particular, $x_{n_0} \in \mathbb{R}^m \setminus S$, which is a contradiction. Hence $x \in S$.

Conversely, let $x \in S$ for every $x \in \mathbb{R}^m$ and for every sequence (x_n) in S with $x_n \to x$. If possible, let S be not closed in \mathbb{R}^m . Then $\mathbb{R}^m \setminus S$ is not open in \mathbb{R}^m and hence there exists $x \in \mathbb{R}^m \setminus S$ such that $x \notin (\mathbb{R}^m \setminus S)^0$. So $B_{1/n}(x) \nsubseteq \mathbb{R}^m \setminus S$ for each $n \in \mathbb{N}$. Hence for each $n \in \mathbb{N}$, there exists $x_n \in B_{1/n}(x)$ such that $x_n \notin \mathbb{R}^m \setminus S$, i.e., $x_n \in S$. We have $||x_n - x|| < \frac{1}{n} \to 0$, which gives $x_n \to x$. Thus we get a contradiction. Therefore S must be closed in \mathbb{R}^m .

Examples:

- (a) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ is a closed set but not an open set in \mathbb{R}^2 .
- (b) More generally, if $x_0 \in \mathbb{R}^m$ and r > 0, then $B_r[x_0]$ is a closed set but not an open set in \mathbb{R}^m .
- (b) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is an open set but not a closed set in \mathbb{R}^2 . More generally, if $x_0 \in \mathbb{R}^m$ and r > 0, then $B_r(x_0)$ is an open set but not a closed set in \mathbb{R}^m .
- (c) $\{(x,y) \in \mathbb{R}^2 : 1 < x < 2\}$ is neither open nor a closed set in \mathbb{R}^2 .
- (d) \mathbb{R}^m is both an open set and a closed set in \mathbb{R}^m .

Theorem: Let S be a nonempty closed and bounded subset of \mathbb{R}^m . If $f: S \to \mathbb{R}^p$ is continuous, then $f(S) = \{f(x) : x \in S\}$ is a closed and bounded subset of \mathbb{R}^p .

Proof: Let $x_n \in S$ for all $n \in \mathbb{N}$ and let $y \in \mathbb{R}^p$ such that $f(x_n) \to y$. Since S is bounded, (x_n) is a bounded sequence in S and hence by the Bolzano-Weierstrass theorem in \mathbb{R}^m , there exist $x_0 \in \mathbb{R}^m$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x_0$. Again, since S is closed in \mathbb{R}^m , $x_0 \in S$. Now, since f is continuous at x_0 , $f(x_{n_k}) \to f(x_0)$. Also, $f(x_n) \to y$ and so $y = f(x_0) \in f(S)$. Therefore f(S) is closed in \mathbb{R}^p .

If possible, let f(S) be not bounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in S$ such that $||f(x_n)|| > n$. Since S is bounded, (x_n) is a bounded sequence in S and hence by the Bolzano-Weierstrass theorem in \mathbb{R}^m , there exist $x_0 \in \mathbb{R}^m$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to x_0$. Again, since S is closed in \mathbb{R}^m , $x_0 \in S$. Now, since f is continuous at x_0 , $f(x_{n_k}) \to f(x_0)$. Thus the sequence $(f(x_{n_k}))$ must be bounded. However, $||f(x_{n_k})|| > n_k$ for all $k \in \mathbb{N}$ and so $(f(x_{n_k}))$ is not bounded. Thus we get a contradiction. Therefore f(S) must be bounded.

Example: We know that $S_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ is a closed and bounded set in \mathbb{R}^2 . Also, \mathbb{R} is not a bounded set in \mathbb{R} and $S_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is not a closed set in \mathbb{R}^2 . Hence there cannot exist any continuous function from S_1 onto \mathbb{R} or onto S_2 .

Theorem: Let S be a nonempty closed and bounded subset of \mathbb{R}^m . If $f: S \to \mathbb{R}$ is continuous, then there exist $x_0, y_0 \in S$ such that $f(x_0) = \sup f(S)$ and $f(y_0) = \inf f(S)$.

Proof: We have proved above that f(S) is a (nonempty) bounded set in \mathbb{R} . Hence $\sup f(S)$, $\inf f(S) \in \mathbb{R}$. Let $\alpha = \sup f(S)$. Then for each $n \in \mathbb{N}$, there exists $x_n \in S$ such that $\alpha - \frac{1}{n} < f(x_n) \le \alpha$. Hence $f(x_n) \to \alpha$. Since $f(x_n) \in f(S)$ for all $n \in \mathbb{N}$ and f(S) is closed in \mathbb{R} (as proved above), $\alpha \in f(S)$. So there exists $x_0 \in S$ such that $\alpha = f(x_0)$. Similarly we can show that there exists $y_0 \in S$ such that $f(y_0) = \inf f(S)$.

Remark: A function $f: S \subseteq \mathbb{R}^m \to \mathbb{R}^p$ is called bounded if f(S) is a bounded subset of \mathbb{R}^p . We note that for a bounded function $f: S \subseteq \mathbb{R}^m \to \mathbb{R}$, it is not necessary that $\max f(S)$ and $\min f(S)$ exist. For example, if $f(x,y) = \frac{x^2+y^2}{1+x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$, then $f: \mathbb{R}^2 \to \mathbb{R}$ is bounded, because $0 \le f(x,y) \le 1$ for all $(x,y) \in \mathbb{R}^2$, but there is no $(x_0,y_0) \in \mathbb{R}^2$ such that $f(x,y) \le f(x_0,y_0)$ for all $(x,y) \in \mathbb{R}^2$.

Limit point: A point $x \in \mathbb{R}^m$ is said to be a limit point of $S \subset \mathbb{R}^m$ if for every r > 0, $(B_r(x) \setminus \{x\}) \cap S \neq \emptyset$ (i.e. if for every r > 0, there exists $y \in S$ such that 0 < ||x-y|| < r). For example, (0,0) and (1,0) are limit points of $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ but (1,1) is not a limit point of S. We note that $(0,0) \in S$ but $(1,0) \notin S$.

Limit: Let x_0 be a limit point of $S \subset \mathbb{R}^m$. Then $y \in \mathbb{R}^k$ is said to be a limit of $f: S \to \mathbb{R}^k$ as x approaches x_0 if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $||f(x) - y|| < \varepsilon$ for all $x \in S$ satisfying $0 < ||x - x_0|| < \delta$.

Such an y, if it exists, is unique because if $z \in \mathbb{R}^k$ with $z \neq y$ is also a limit of f as x approaches x_0 , then $\varepsilon = \frac{1}{2}||y-z|| > 0$ and so there exist $\delta_1, \delta_2 > 0$ such that $f(x) \in B_{\varepsilon}(y)$ for all $x \in (B_{\delta_1}(x_0) \setminus \{x_0\}) \cap S$ and $f(x) \in B_{\varepsilon}(z)$ for all $x \in (B_{\delta_2}(x_0) \setminus \{x_0\}) \cap S$. If $\delta = \min\{\delta_1, \delta_2\}$, then $\delta > 0$ and since $(B_{\delta}(x_0) \setminus \{x_0\}) \cap S \neq \emptyset$, we can choose $x \in (B_{\delta}(x_0) \setminus \{x_0\}) \cap S$. Then $f(x) \in B_{\varepsilon}(y) \cap B_{\varepsilon}(z)$, which contradicts the fact that $B_{\varepsilon}(y) \cap B_{\varepsilon}(z) = \emptyset$, proved earlier.

The y appearing in the above definition is called the limit of f as x approaches x_0 and we write

$$\lim_{x \to x_0} f(x) = y.$$

Sequential criterion of limit: Let x_0 be a limit point of $S \subset \mathbb{R}^m$ and let $f: S \to \mathbb{R}^k$.

$$\lim_{x \to x_0} f(x) = y \in \mathbb{R}^k \iff \text{for every sequence } (x_n) \text{ in } S \setminus \{x_0\} \text{ converging to } x_0,$$

the sequence
$$(f(x_n))$$
 converges to y .

Proof: Let $\lim_{x\to x_0} f(x) = y$ and let (x_n) be a sequence in $S\setminus\{x_0\}$ such that $x_n\to x_0$. If $\varepsilon > 0$, then there exists $\delta > 0$ such that $\|f(x) - y\| < \varepsilon$ for all $x \in S$ satisfying $0 < \|x - x_0\| < \delta$. Also, since $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$, there exists $n_0 \in \mathbb{N}$ such that $0 < \|x_n - x_0\| < \delta$ for all $n \ge n_0$. Hence $\|f(x_n) - y\| < \varepsilon$ for all $n \ge n_0$. Thus $(f(x_n)) \to y$.

Conversely, let $f(x_n) \to y$ for every sequence (x_n) in $S \setminus \{x_0\}$ with $x_n \to x_0$. If possible, let $\lim_{x \to x_0} f(x) \neq y$. Then there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there exists $x_n \in S$ with $0 < \|x_n - x_0\| < \frac{1}{n}$ and $\|f(x_n) - y\| \ge \varepsilon$. Thus (x_n) is a sequence in $S \setminus \{x_0\}$ such that $\|x_n - x_0\| \to 0$, i.e. $x_n \to x_0$ but $f(x_n) \not\to y$. This is a contradiction. Therefore $\lim_{x \to x_0} f(x) = y$.

Examples:

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2} = 0$$

but

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$
 does not exist (in \mathbb{R}).

A method for showing the non-existence of limit: Let (x_0, y_0) be a limit point of $S \subset \mathbb{R}^2$ and let $f: S \to \mathbb{R}$. Let $D \subseteq \mathbb{R}$ such that x_0 is a limit point of D and let $\varphi: D \to \mathbb{R}$ such

that $(x,\varphi(x)) \in S$ for all $x \in D$ and $\lim_{x\to x_0} \varphi(x) = y_0$. If $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \in$ \mathbb{R} , then

$$\lim_{x \to x_0} f(x, \varphi(x)) = \ell.$$

Proof: Let (x_n) be any sequence in $D\setminus\{x_0\}$ such that $x_n\to x_0$. Since $\lim_{x\to x_0}\varphi(x)=y_0$, we get $\varphi(x_n)\to y_0$. Now, $((x_n,\varphi(x_n)))$ is a sequence in $S\setminus\{(x_0,y_0)\}$ and $((x_n,\varphi(x_n)))\to y_0$. (x_0, y_0) . Since $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$, we have $f(x_n, \varphi(x_n)) \to \ell$. Consequently

$$\lim_{x \to x_0} f(x, \varphi(x)) = \ell.$$

Examples:

(a) If $f(x,y) = \frac{xy}{x^2 + y^2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, then

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist (in \mathbb{R}) because if $m \in \mathbb{R}$, then $\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$,

which gives more than one value if we vary m. (b) If $f(x,y) = \frac{x^2y}{x^4+y^2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in \mathbb{R}) because if $m \in \mathbb{R}$, then $\lim_{x\to 0} f(x, mx^2) = \lim_{x\to 0} \frac{mx^4}{x^4 + m^2x^4} = \frac{m}{1+m^2}$ for all $m \in \mathbb{R}$.

Remark: The polar coordinates in \mathbb{R}^2 can also be used in the evaluation of certain limits of functions $f: S \subseteq \mathbb{R}^2 \to \mathbb{R}$. For example, taking $x = r \cos \theta, y = r \sin \theta$, where r>0 and $\theta\in[0,2\pi)$, we find that

$$\left| \frac{x^2}{x^2 + y^2} \right| = |r| \cos^2 \theta$$

 $|s| = r \to 0$ as $r \to 0$ and hence we can conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} = 0.$$

However, while using this method we should not assume θ to be a constant while taking limit as $r \to 0$.

Theorem: Let x_0 be a limit point of $S \subseteq \mathbb{R}^m$ and let $f: S \to \mathbb{R}^k$. If f(x) = $(f_1(x_1),...,f_k(x_k))$ for all $x \in S$, where $f_j: S \to \mathbb{R}$ for j=1,...,k, then $\lim_{x\to x_0} f(x)$ exists in \mathbb{R}^k iff $\lim_{x\to x_0} f_j(x)$ exists in \mathbb{R} for each $j\in\{1,...,k\}$, and in that case $\lim_{x\to x_0} f(x)=$ $(\lim_{x\to x_0} f_1(x), ..., \lim_{x\to x_0} f_k(x)).$

Proof: Let us first assume that $\lim_{x\to x_0} f(x) = y = (y_1, ..., y_k) \in \mathbb{R}^k$ and let (x_n) be any sequence in $S \setminus \{x_0\}$ such that $x_n \to x_0$. Then $f(x_n) = (f_1(x_n), ..., f_k(x_n)) \to (y_1, ..., y_k)$ and hence $f_j(x_n) \to y_j$ for each $j \in \{1, ..., k\}$. Consequently $\lim_{x\to x_0} f_j(x) = y_j$ for each $j \in \{1, ..., k\}$.

Conversely, let $\lim_{x\to x_0} f_j(x) = y_j \in \mathbb{R}$ for each $j \in \{1, ..., k\}$ and let (x_n) be any sequence in $S \setminus \{x_0\}$ such that $x_n \to x_0$. Then $f_j(x_n) \to y_j$ for each $j \in \{1, ..., k\}$ and hence $f(x_n) = (f_1(x_n), ..., f_k(x_n)) \to (y_1, ..., y_k) \in \mathbb{R}^k$. Therefore $\lim_{x\to x_0} f(x) = (y_1, ..., y_k) \in \mathbb{R}^k$.

Remark: The limit rules for combinations of functions can be given similar to those for continuity.

Relation between limit and continuity: Let $S \subseteq \mathbb{R}^m$ and $x_0 \in S$.

If x_0 is also a limit point of S, then from the definitions of continuity and limit, it follows immediately that $f: S \to \mathbb{R}^k$ is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$.

On the other hand, if x_0 is not a limit point of S, then there exists $\delta > 0$ such that $B_{\delta}(x_0) \cap S = \{x_0\}$ and so $||f(x) - f(x_0)|| = 0$ for all $x \in S$ satisfying $||x - x_0|| < \delta$. Consequently f is continuous at x_0 .

Infinite limits: Let x_0 be a limit point of $S \subseteq \mathbb{R}^n$ and let $f: S \to \mathbb{R}$. Then we write $\lim_{x\to x_0} f(x) = \infty$ if for every r > 0, there exists $\delta > 0$ such that f(x) > r for all $x \in S$ satisfying $||x - x_0|| < \delta$.

It can be shown that $\lim_{x\to x_0} f(x) = \infty$ if and only if for every sequence (x_n) in $S\setminus \{x_0\}$ converging to $x_0, f(x_n)\to \infty$.

We can also define $\lim_{x\to x_0} f(x) = -\infty$ analogously and obtain its sequential criterion.

Example:

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2 + y^2} = \infty \quad \text{but} \quad \lim_{(x,y)\to(0,0)} \frac{1}{x + y} \neq \infty.$$

Differentiability of vector valued function of one real variable: A function $F: S \subset \mathbb{R} \to \mathbb{R}^k$ is said to be differentiable at $t_0 \in S^0$ if $\lim_{t \to t_0} \frac{1}{t - t_0} (F(t) - F(t_0))$ exists (in \mathbb{R}^k) and in that case the derivative of F at t_0 is defined as $F'(t_0) = \lim_{t \to t_0} \frac{1}{t - t_0} (F(t) - F(t_0))$. If $F(t) = (f_1(t), \ldots, f_k(t))$ for all $t \in S$, then F is differentiable at t_0 ($t_0 \in S^0$) if and only if $f_j: S \to \mathbb{R}$ is differentiable at t_0 for each $j \in \{1, \ldots, k\}$, and in such case $F'(t_0) = (f'_1(t_0), \ldots, f'_k(t_0))$.

We say that $F: S \subset \mathbb{R} \to \mathbb{R}^k$ is differentiable (on S) if F is differentiable at every point of S.

Example: If $F(t) = (\cos t, \sin t, t)$ for all $t \in \mathbb{R}$, then $F : \mathbb{R} \to \mathbb{R}^3$ is differentiable (since each component function of F is differentiable) and $F'(t) = (-\sin t, \cos t, 1)$ for all $t \in \mathbb{R}$.

Remark: Let $F: S \subset \mathbb{R} \to \mathbb{R}^k$ and let $t_0 \in S \setminus \partial S$ be one point of an interval contained in S. Then the differentiability and the derivative of F at t_0 are defined as in the above definition by considering $t \to t_0^+$ or $t \to t_0^-$, whichever is applicable.

Differentiation of composite functions: Differentiable functions can be combined (in meaningful ways) to produce new differentiable functions. We illustrate this with the following results. Let $F, G: S \subset \mathbb{R} \to \mathbb{R}^k$ be differentiable at $t_0 \in S^0$. Then

- (a) $F + G : S \to \mathbb{R}^k$ is differentiable at t_0 and $(F + G)'(t_0) = F'(t_0) + G'(t_0)$.
- (b) If $F: G: S \to \mathbb{R}^k$ is differentiable at t_0 and $(F \cdot G)'(t_0) = F'(t_0) \cdot G(t_0) + F(t_0) \cdot G'(t_0)$.
- (c) $\varphi F: S \to \mathbb{R}^k$ is differentiable at t_0 and $(\varphi F)'(t_0) = \varphi'(t_0)F(t_0) + \varphi(t_0)F'(t_0)$, where $\varphi: S \to \mathbb{R}$ is differentiable at t_0 .

We prove (b). The other two can be similarly proved.

Proof of (b): Let $F(t) = (f_1(t), \ldots, f_k(t))$ and $G(t) = (g_1(t), \ldots, g_k(t))$ for all $t \in S$, where $f_j, g_j : S \to \mathbb{R}$ for each $j \in \{1, \ldots, k\}$. Then $(F \cdot G)(t) = F(t) \cdot G(t) = \sum_{j=1}^k f_j(t)g_j(t)$ for all $t \in S$.

Since F and G are differentiable at t_0 , f_j and g_j are differentiable at t_0 for each $j \in \{1, \ldots, k\}$ and hence $F \cdot G$ is differentiable at t_0 . Also,

$$(F \cdot G)'(t_0) = \sum_{j=1}^k f_j'(t_0)g_j(t_0) + \sum_{j=1}^k f_j(t_0)g_j'(t_0) = F'(t_0) \cdot G(t_0) + F(t_0) \cdot G'(t_0).$$

Chain rule: Let $\varphi: D \subset \mathbb{R} \to \mathbb{R}$ and $F: S \subset \mathbb{R} \to \mathbb{R}^k$ be such that $\varphi(D) \subset S$. If φ is differentiable at $s_0 \in D^0$ and F is differentiable at $t_0 = \varphi(s_0) \in S^0$, then $F \circ \varphi: D \to \mathbb{R}^k$ is differentiable at s_0 and $(F \circ \varphi)'(s_0) = \varphi'(s_0)F'(\varphi(s_0))$.

Proof: Let $F(t) = (f_1(t), \ldots, f_k(t))$ for all $t \in S$, where $f_j : S \to \mathbb{R}$ for each $j \in \{1, \ldots, k\}$. Since F is differentiable at t_0 , f_j is differentiable at t_0 for each $j \in \{1, \ldots, k\}$. Hence by the chain rule of calculus of one real variable, $f_j \circ \varphi : D \to \mathbb{R}$ is differentiable at s_0 and $(f_j \circ \varphi)'(s_0) = \varphi'(s_0)f'_j(\varphi(s_0))$. Thus $F \circ \varphi$ is differentiable at s_0 and

$$(F \circ \varphi)'(s_0) = (\varphi'(s_0)f_1'(\varphi(s_0)), \dots, \varphi'(s_0)f_k'(\varphi(s_0))) = \varphi'(s_0)F'(\varphi(s_0)).$$

for each $j \in \{1, ..., k\}$. Since

$$(F \circ \varphi)(s) = F(\varphi(s)) = (f_1(\varphi(s)), \dots, f_k(\varphi(s))) = ((f_1 \circ \varphi)(s), \dots, (f_k \circ \varphi)(s))$$

for all $s \in D$, $F \circ \varphi$ is differentiable at s_0 and $(F \circ \varphi)'(s_0) = (f_1 \circ \varphi)'(s_0), \dots, (f_k \circ \varphi)'(s_0)$

$$= (\varphi'(s_0)f_1'(t_0), \dots, \varphi'(s_0)f_k'(t_0)) = \varphi'(s_0)(f_1'(t_0), \dots, f_k'(t_0)) = \varphi'(s_0)F'(t_0).$$

Example: Let $\varphi(s) = 2s^4 + 3s - 3$ for all $s \in \mathbb{R}$ and $F(t) = (2t^3, t^6 + 9, 5t^4 + 1)$ for all $t \in \mathbb{R}$. Then $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable at 1 and $F : \mathbb{R} \to \mathbb{R}^3$ is differentiable at $\varphi(1) = 2$. Hence by chain rule, $F \circ \varphi : \mathbb{R} \to \mathbb{R}^3$ is differentiable at 1 and

$$(F \circ \varphi)'(1) = \varphi'(1)F'(2) = 11(24, 192, 160) = (264, 2112, 1760).$$

However, since $(F \circ \varphi)(s) = (2(\varphi(s))^3, (\varphi(s))^6 + 9, 5(\varphi(s))^4 + 1)$ for all $s \in \mathbb{R}$, in this case without using the above chain rule also we can directly obtain that $F \circ \varphi$ is differentiable at 1 and

$$(F \circ \varphi)'(1) = (264, 2112, 1760).$$

Partial derivative: Let $f: S \subset \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in S^0$. The partial derivative of f with respect to x (the first variable) at (x_0, y_0) is defined as

$$\left. \frac{\partial f}{\partial x}(x_0, y_0) = \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} = f_x(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t},$$

provided this limit exists (in \mathbb{R}).

Similarly, the partial derivative of f with respect to y (the second variable) at (x_0, y_0) is defined as

$$\left. \frac{\partial f}{\partial y}(x_0, y_0) = \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} = f_y(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0, y_0 + t) - f(x_0, y_0)}{t},$$

provided this limit exists (in \mathbb{R}).

Thus if $f: \{x \in \mathbb{R} : (x,y_0) \in S\}$, $B = \{y \in \mathbb{R} : (x_0,y) \in S\}$, $\varphi(x) = f(x,y_0)$ for all $x \in A$ and $\psi(y) = f(x_0,y)$ for all $y \in B$, then $f_x(x_0,y_0) = \varphi'(x_0)$ and $f_y(x_0,y_0) = \psi'(y_0)$. More generally, if $f: S \subset \mathbb{R}^m \to \mathbb{R}$, $x_0 \in S^0$ and $j \in \{1,\ldots,m\}$, then the partial derivative of f with respect to x_j at x_0 is

$$\left. \frac{\partial f}{\partial x_j}(x_0) = \left. \frac{\partial f}{\partial x_j} \right|_{x_0} = f_{x_j}(x_0) = \lim_{t \to 0} \frac{f(x_0 + te_j) - f(x_0)}{t},$$

provided this limit exists (in \mathbb{R}).