MA15010H: Multi-variable Calculus

(Lecturenote 5: Riemann Integration, Fubini's Theorem)
September - November, 2025

1. A revision of Riemann integral of one variable

Let $f : [a, b] \to \mathbb{R}$ be bounded function and $P = \{x_o, x_1, \dots, x_n\}$ is a partition of $[a, b]$, where $\{a = x_o < x_1 < \cdots < x_n = b\}$. Let $\Delta x_i = x_i - x_{i-1}$. Define $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$ Write

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i \text{ and } U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Since f is bounded, there exist $m, M \ge 0$ such that $m \le f(x) \le M$ for all $x \in [a, b]$. Hence

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a).$$

It is easy to see that if $P_1 \subseteq P_2$, then $U(P_1, f) \ge U(P_2, f)$ and $L(P_1, f) \le L(P_2, f)$. It is clear that $L(P, f)$ is an increasing function over the set of all finer partitions while $U(P, f)$ is a decreasing function of P.

Definition 1.1. The function f is said to be Riemann integrable (or $f \in \mathcal{R}[a, b]$) if

$$\inf_P U(P, f) = \sup_P L(P, f).$$

Let $\omega(P, f) = U(P, f) - L(P, f)$. From the definition, it follows that

$$\inf_P \omega(P, f) = \inf_P \{U(P, f) - L(P, f)\} = 0, \tag{1.1}$$

where $\omega(P, f)$ is known as oscillatory sum of f over the partition P. Hence, if $f \in \mathcal{R}[a, b]$, then for each $\epsilon > 0$, there exists a partition P such that $\omega(P, f) < \epsilon$. On the other hand, for $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}$, there exists a partition P_n such that $\omega(P_n, f) < \frac{1}{n}$. Thus, $\lim_{n\to\infty} \omega(P_n, f) = 0$.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a, b]$ if and only if there exists a sequence $\{P_n\}$ of partitions of $[a, b]$ such that $\lim_{n\to\infty} \omega(P_n, f) = 0$.*

Proof. We have already seen the forward implication. For the other one, if $\lim_{n\to\infty} \omega(P_n, f) = 0$, then for each $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ such that $\omega(P_n, f) < \epsilon$, whenever $n \ge n_o$. But, then $\inf_P \omega(P, f) \le \omega(P_{n_o}, f) < \epsilon$ for all $\epsilon > 0$. Since f is bounded, both $\inf_P U(P, f)$ and $\sup_P L(P, f)$ exist, and from (1.1) it follows that $\inf_P U(P, f) = \sup_P L(P, f)$. Hence $f \in \mathcal{R}[a, b]$. □

Example 1.3. Let $f : [0, 1] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is bounded and for $P_n = \{\frac{i}{n} : \ i = 0, 1, \dots, n\}$, we have

$$\omega(P_n, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \le 2.\frac{1}{n} \to 0,$$

since $\frac{1}{2}$ can belong to two consecutive subintervals. Hence $f \in \mathcal{R}[0, 1]$.

Recall that if $P_1 \subseteq P_2$, then $U(P_1, f) \geq U(P_2, f)$ and $L(P_1, f) \leq L(P_2, f)$. Hence $\omega(P_1, f) \geq \omega(P_2, f)$. Using this fact, it is enough to workout $\lim_{n\to\infty} \omega(P_n, f) = 0$, while $\{P_n\}$ is an increasing sequence of partitions.

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a, b]$ if and only if there exists an increasing sequence of partitions $\{P_n\}$ of $[a, b]$ such that $\lim_{n\to\infty} \omega(P_n, f) = 0$.*

Proof. Since $f \in \mathcal{R}[a, b]$, by Theorem 1.2, there exists a sequence of partition $\{P_n\}$ such that $\lim_{n\to\infty} \omega(P_n, f) = 0$. Let $Q_1 = P_1$ and $Q_n = P_1 \cup P_2 \cup \cdots \cup P_n$. Then $\omega(Q_n, f) \leq \omega(P_n, f) \to 0$. The converse part is obvious from Theorem 1.2. □

Remark 1.5. From Theorem 1.4 it follows that $\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = \int_a^b f(x)dx$.

Theorem 1.6. *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}[a, b]$.*

Proof. Since f is continuous on the closed interval $[a, b]$, f is bounded and uniformly continuous. For each $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$. Choose a partition P of $[a, b]$ such that $\Delta x_i < \delta$. Since f attains its infimum and supremum on each subinterval, we get $M_i - m_i \leq \frac{\epsilon}{2(b-a)}$. Hence

$$\omega(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq \sum_{i=1}^{n} \frac{\epsilon}{2(b-a)} \Delta x_i < \epsilon.$$

□

Example 1.7. Every monotone function f on $[a, b]$ is Riemann integrable. Assume f is monotone increasing. Let $P_n = \left\{x_i = a + \frac{(b-a)i}{n} : \ i = 0, 1, \ldots, n\right\}$. Then the oscillatory sum

$$\omega(P_n, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\} \frac{b-a}{n} = \{f(b) - f(a)\} \frac{b-a}{n} \to 0.$$

Hence by Theorem 1.4 we conclude that $f \in \mathcal{R}[a, b]$.

Continuity like condition for Riemann integrability on $[a, b]$.

We know that the oscillatory sum $\omega(P, f)$ decreases over the set of finer partitions. And f is Riemann integrable if and only if there is a sequence of partitions $\{P_n\}$ such that $\omega(P_n, f) \to 0$. Using this fact, we derive a continuity like condition for Riemann integrability of bounded function on $[a, b]$. For a given partition $P = \{x_o, x_1, \ldots, x_n\}$ of $[a, b]$, we define $|P| = \max_{1\leq i\leq n} \Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$.

Theorem 1.8. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}([a, b])$ if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each partition P with $|P| < \delta$ implies $\omega(P, f) < \epsilon$.*

Proof. Since f is Riemann integrable, for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $\omega(P, f) < \epsilon$. Let $\delta > 0$ be small enough and P' be a refinement of P such that $|P'| < \delta$. As $P \subseteq P'$, it follows that $\omega(P', f) \leq \omega(P, f) < \epsilon$. The other implication is obvious by definition of $\mathcal{R}([a, b])$. □

Corollary 1.9. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}([a, b])$ if and only if for each sequence of partitions $\{P_n\}$ with $|P_n| \to 0$ implies $\omega(P_n, f) \to 0$.*

Question*. Think about, how far can be a Riemann integrable function from continuous function.

Double integrals

We know that the Riemann integral of a non-negative function of one variable on a finite interval is the area of the region under the graph of the function. In a similar way, the double integral of a non-negative function $f(x, y)$ defined on a region in the plane is the volume of the region under the graph of $f(x, y)$.

First, we discuss double integral on the rectangular region, and later we consider more general region with curvilinear boundary.

Let $D = [a, b] \times [c, d]$ and $f : D \to \mathbb{R}$ be bounded. Let $P_1 = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and $P_2 = \{y_0, y_1, \ldots, y_m\}$ be a partition of $[c, d]$. Note that the partition $P = P_1 \times P_2$ decomposes D into mn sub-rectangles (or cells). Let $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Let $m_{ij} = \inf\{f(x, y) : (x, y) \in D_{ij}\}$. Define

$$L(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} \Delta x_i \Delta y_j.$$

Similarly, we can define

$$U(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} \Delta x_i \Delta y_j,$$

where $M_{ij} = \sup\{f(x, y) : (x, y) \in D_{ij}\}$. The lower integral of f is defined by $\sup_P L(P, f)$. The upper integral of f is defined by $\inf_P U(P, f)$. Note that both the integrals exist because f is bounded. We say that f is integrable on D (or $f \in \mathcal{R}(D)$) if both lower and upper integrals of f are equal. If the function f is integrable on D, then the **double integral** is denoted by

$$\iint_D f(x, y)dxdy \text{ or } \iint_D f(x, y)dA.$$

Example 1.10. Let $f : D = [0, 1] \times [0, 1] \to \mathbb{R}$ is given by

$$f(x, y) = \begin{cases} 1 & \text{if } x, y \in \mathbb{Q} \cap [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Then f is not integrable on D, because for any partition P of D defined as above, we get $U(P, f) = 1 \neq 0 = L(P, f)$.

Theorem 1.11. *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}(D)$ if and only if for each $\epsilon > 0$ there exists a partition P of D such that $\omega(P, f) = U(P, f) - L(P, f) < \epsilon$.*

Theorem 1.12. *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}(D)$ if and only if there exists an increasing sequence of partitions $\{P_n\}$ of D such that $\lim_{n\to\infty} \omega(P_n, f) = 0$.*

Since the proof of Theorem 1.12 is similar to Theorem 1.4, we omit here.

Example 1.13. Let $f : D = [0,1] \times [0,1] \to \mathbb{R}$ is given by
$$f(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$
Then $\iint\limits_D f(x,y)dxdy = 0$. Let $P_n = \{\frac{i}{n} : \ i = 0,1,\ldots,n\} \times \{\frac{i}{n} : \ i = 0,1,\ldots,n\}$. In this case, $\Delta x_i = \Delta y_j = \frac{1}{n}$. The oscillatory sum of the function f on D satisfies
$$\omega(P_n, f) = \sum_{i=1}^{n}\sum_{j=1}^{n}(M_{ij} - m_{ij})\Delta x_i \Delta y_j = \sum_{i=1}^{n}\sum_{j=1}^{n}(M_{ij} - 0)\frac{1}{n^2} = \sum_{i=j,\ j=1}^{n} 1.\frac{1}{n^2} = \frac{1}{n} \to 0.$$

Theorem 1.14. *Let $D = [a,b] \times [c,d]$. If $f : D \to \mathbb{R}$ is continuous, then f is integrable on D.*

Proof. Since f is continuous on the closed rectangle D, it follows that f is bounded and uniformly continuous on D. Hence for given $\epsilon > 0$ there exists $\delta >$ such that for $(x,y),\ (x',y') \in D$ with $\sqrt{(x-x')^2 + (y-y')^2} < \delta$ implies $|f(x,y) - f(x',y')| < \frac{\epsilon}{2A}$, where A is the area of the rectangle D. Let $P = \{D_{ij} : \ i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\}$, where $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Write $d(D_{ij}) = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{j-1})^2}$. Now, suppose P satisfies $d(D_{ij}) < \delta$ for all $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$. Since f attains its infimum and supremum on each closed cell D_{ij}, we get $M_{ij} - m_{ij} \leq \frac{\epsilon}{2A}$. Hence
$$\omega(P,f) = \sum_{i=1}^{n}\sum_{i=1}^{m}(M_{ij} - m_{ij})\Delta x_i \Delta y_i \leq \sum_{i=1}^{n}\sum_{i=1}^{m}\frac{\epsilon}{2A}\Delta x_i \Delta y_j < \epsilon.$$
Hence by Theorem 1.12, we conclude that $f \in \mathcal{R}(D)$. □

Continuity like condition for Riemann integrability

Let $P = \{D_{ij} : \ i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\}$ be a partition of $D = [a,b] \times [c,d]$, where $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Write $d(D_{ij}) = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$. Define $|P| = \max\{d(D_{ij}) : \ i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\}$.

Theorem 1.15. *Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}(D)$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each partition P of D with $|P| < \delta$ implies $\omega(P,f) < \epsilon$.*

Since the proof of Theorem 1.15 is similar to Theorem 1.8, we omit here.

Corollary 1.16. *Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}(D)$ if and only if for each sequence of partitions $\{P_n\}$ of D with $|P_n| \to 0$ implies $\omega(P_n, f) \to 0$.*

Note that in order to show $f \notin \mathcal{R}(D)$, it is enough to show that there exists a sequence of partitions $\{P_n\}$ with $|P_n| \to 0$ but $\omega(P_n, f) \not\to 0$.

Geometric Interpretation

If $f : D = [a,b] \times [c,d] \to [0,\infty)$ is integrable. Then $\iint\limits_D f(x,y)dxdy$ is the volume of the region bounded by planes $x = a,\ x = b,\ y = c,\ y = d$ and the surface $z = f(x,y)$.

Repeated Integrals. The next result illustrates that the evaluation of the double integral can be reduced to the repeated integrals. This result is known as Fubini's Theorem. Before we come to the main result let us have a look at the following examples.

Example 1.17. Consider $f : D = [0,1] \times [0,1] \to \mathbb{R}$, defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 2y, & \text{if } x \in \mathbb{Q}^c \cap [0,1] \end{cases}.$$

Then $\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = 1$. However f is not integrable on D. (**Hint:** Use Corollary 1.16 to deduce that $f \notin \mathcal{R}(D)$.)

Example 1.18. Consider $f : D = [0,1] \times [0,1] \to \mathbb{R}$, defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2},\ y \in \mathbb{Q} \cap [0,1] \\ -1 & \text{if } x = \frac{1}{2},\ y \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Note that for $x = \frac{1}{2}$, $\int_0^1 f(x,y)dy$ does not exists. However, $\iint\limits_D f(x,y)dxdy$ exists.

Theorem 1.19. (Fubini's Theorem) Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be integrable. If for each $y \in [c,d]$, the function $f(\cdot, y) \in \mathcal{R}[a,b]$, then the function F defined by $F(y) = \int\limits_a^b f(x,y)dx$ is integrable on $[c,d]$ and

$$\iint\limits_D f(x,y)dxdy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

Proof. Since $f \in \mathcal{R}(D)$, for each $\epsilon > 0$ there exists a partition

$$P = P_1 \times P_2 = \{D_{ij} : \ i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,m\}$$

of D such that $U(P,f) - L(P,f) < \epsilon$. Recall that $m_{ij} = \inf\{f(x,y) : \ (x,y) \in D_{ij}\}$ and $M_{ij} = \sup\{f(x,y) : \ (x,y) \in D_{ij}\}$. Let us define $k_j = \inf\{F(y) : \ y_{j-1} \leq y \leq y_j\}$ and $K_j = \sup\{F(y) : \ y_{j-1} \leq y \leq y_j\}$. Since $m_{ij} \leq f(x,y) \leq M_{ij}$ for each $(x,y) \in D_{ij}$, it follows that

$$\sum_{i=1}^{n} m_{ij}\Delta x_i \leq L(P_1, f(\cdot,y)) \leq \int_a^b f(x,y)dx = F(y) \leq \sum_{i=1}^{n} M_{ij}\Delta x_i \tag{1.2}$$

for each $y \in [y_{j-1}, y_j]$. Note the **first inequality** in (1.2) follows due to the fact that infimum m_{ij} of f on D_{ij} is smaller than the infimum of f over $[x_{i-1}, x_i] \times \{y\}$.

From the above it follows that

$$L(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\Delta x_i \Delta y_j \leq \sum_{j=1}^{n} k_j \Delta y_j = L(P_2, F) \leq U(P_2, F)$$

and

$$U(P_2, F) = \sum_{j=1}^{n} K_j \Delta y_j \leq \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}\Delta x_i \Delta y_j = U(P,f).$$

Hence

$$L(P,f) \leq L(P_2,F) \leq U(P_2,F) \leq U(P,f). \tag{1.3}$$

Since $U(P,f)-L(P,f)<\epsilon$, from (1.3) we get $U(P_2,F)-L(P_2,F)<\epsilon$. That is, $F\in\mathcal{R}[c,d]$, and hence once again from (1.3) we infer that

$$L(P,f)\le\int_c^d F(y)dy\le U(P,f) \text{ and } L(P,f)\le\iint\limits_D f(x,y)dxdy\le U(P,f).$$

Thus,

$$-\epsilon<\iint\limits_D f(x,y)dxdy-\int_c^d F(y)dy\le\epsilon$$

for each $\epsilon>0$. Hence

$$\iint\limits_D f(x,y)dxdy=\int_c^d F(y)dy.$$

This completes the proof. □

Note that if we define $G(x)=\int\limits_c^d f(x,y)dy$, then the similar result holds.

Corollary 1.20. (Fubini's Theorem) Let $f:D=[a,b]\times[c,d]\to\mathbb{R}$ be a continuous function. Then

$$\iint\limits_D f(x,y)dxdy=\int_c^d\left(\int_a^b f(x,y)dx\right)dy=\int_a^b\left(\int_c^d f(x,y)dy\right)dx.$$

Example 1.21. Let $f(x,y)=xe^{xy}$ for $(x,y)\in D=[0,2]\times[0,1]$. Then f is continuous and hence by Fubini's theorem

$$\iint\limits_D f(x,y)dxdy=\int_0^2\left(\int_0^1 xe^{xy}dy\right)dx=\int_0^2\left[e^{xy}\right]_0^1dx=\int_0^2(e^x-1)dx=e^2-3.$$

Bounded functions with discontinuities. We know from Theorem 1.14 that if f is continuous on D then f is integrable. In this section, we discuss that the integral of a function f also exists if the set of discontinuities of f is not too large. In order to measure discontinuities, we introduce the following concept.

Definition 1.22. Let A be a bounded subset of $\mathbb{R}^2$. Then A is said to be of content zero if for each $\epsilon>0$ there exist finitely many rectangles $\{R_i\}_{i=1}^n$ such that $A\subseteq\bigcup\limits_{i=1}^n R_i$ and $\text{Area}\left(\bigcup\limits_{i=1}^n R_i\right)<\epsilon$.

Example 1.23. (i) Any finite set of points in $\mathbb{R}^2$ has content zero.
(ii) Every subset of a set of content zero has content zero.
(iii) The union of finite numbers of bounded sets of content zero is also of content zero.
(iv) Every line segment has content zero.

Exercise 1.24. Any bounded subset of $\mathbb{R}^2$ having non-empty interior cannot have content zero.

Theorem 1.25. *Let $f:D=[a,b]\times[c,d]\to\mathbb{R}$ be a bounded function. If the set of discontinuities of f in D is a set of content zero, then f is integrable.*

Proof. Let $M > 0$ be such that $|f(x,y)| \le M$ for all $(x,y) \in D$. Suppose E is set of discontinuities of f in D. In order to prove this result, we need to reorganize some symbols. Let $P = \{D_i : D_i \text{ subrectangles in D}\}$ be a partition of D. Let $m_i = \inf_{D_i}(f),\ M_i = \sup_{D_i}(f)$ and $A(D_i) = \text{Area}(D_i)$. Now, choose a partition P of D such that

$$E \subset \bigcup_{i=1}^{m} D_i \text{ and } \sum_{i=1}^{m} A(D_i) < \frac{\epsilon}{4M}.$$

Note that f is uniformly continuous on each closed subrectangle $D_i : i = m+1, \dots, n$. Hence f attains its infimum and supremum on each D_i. Thus, as similar argument used in the proof of Theorem 1.14, we can have selected the partition P such that $M_i - m_i \le \frac{\epsilon}{2A(D)}$ for $i = m+1, \dots, n$. Hence

$$\begin{aligned}
\omega(P,f) &= \sum_{i=1}^{n}(M_i - m_i)A(D_i) \\
&= \sum_{i=1}^{m}(M_i - m_i)A(D_i) + \sum_{i=m+1}^{n}(M_i - m_i)A(D_i) \\
&\le \sum_{i=1}^{m} 2MA(D_i) + \sum_{i=m+1}^{n} \frac{\epsilon}{2A(D)}A(D_i) \\
&< 2M\frac{\epsilon}{4M} + \frac{\epsilon}{2}\frac{A(D)}{A(D)} = \epsilon.
\end{aligned}$$

Thus, for each $\epsilon > 0$ we have constructed a partition P of D such that $\omega(P,f) < \epsilon$. This implies $f \in \mathcal{R}(D)$. □

Double integral over general bounded regions. Let D be a bounded region in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a bounded function defined on D. Let Q be a rectangle such that $D \subseteq Q$. Extend f on Q as $\tilde{f} : Q \to \mathbb{R}$, where

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in Q \setminus D. \end{cases}$$

If $\tilde{f}$ is integrable over Q, then we say that f is integrable over D and define

$$\iint_D f(x,y)dxdy = \iint_Q \tilde{f}(x,y)dxdy.$$

Theorem 1.26. (Fubini's Theorem) Let f be a bounded continuous function over a bounded region D in $\mathbb{R}^2$.

(i) If $D = \{(x,y) : a \le x \le b \text{ and } f_1(x) \le y \le f_2(x)\}$ for some continuous functions $f_1, f_2 : [a,b] \to \mathbb{R}$, then

$$\iint_D f(x,y)dxdy = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)dy \right) dx.$$

(ii) If $D = \{(x,y) : c \leq y \leq d \text{ and } g_1(y) \leq x \leq g_2(y)\}$ for some continuous functions $g_1, g_2 : [c,d] \to \mathbb{R}$, then

$$\iint_D f(x,y)dxdy = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)dx \right) dy.$$

For a proof of Theorem 1.26, we refer to Chapter 11, Calculus Vol. II, by Apostol.

Example 1.27. (i) Let D be the region bounded by the lines joining the points $(0,0)$, $(0,1)$ and $(2,2)$. Evaluate the integral $\iint_D (x+y)^2 dxdy$.

(ii) Evaluate the integral $\int_0^2 \left(\int_{\frac{y}{2}}^1 e^{x^2} dx \right) dy$.

Riemann integrable functions on D satisfy the following algebraic relations.

Theorem 1.28. *Let f and g be Riemann integrable functions on the region D in the plane and $c \in \mathbb{R}$. Then*

(i) $cf + g \in R(D)$, $\iint_D \{cf(x,y) + g(x,y)\}dxdy = c\iint_D f(x,y)dxdy + \iint_D g(x,y)dxdy.$

(ii) *If $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then $\iint_D f(x,y)dxdy \leq \iint_D g(x,y)dxdy$.*

(iii) $|f| \in \mathcal{R}(D)$ *and* $\left| \iint_D f(x,y)dxdy \right| \leq \iint_D |f(x,y)|dxdy.$