MA15010H: Multi-variable Calculus

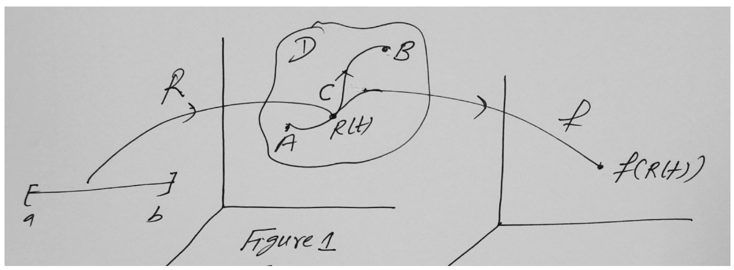
(Lecturenote 7: Line and surface integrals)

September - November, 2025

Let $R : [a, b] \to \mathbb{R}^3$ be a differentiable function and the curve C is parameterized by $R(t)$. Suppose $f : C \to \mathbb{R}^3$ is a bounded function. The line integral of f along C is denoted by the symbol $\int\limits_C f \cdot dR$ and is defined by

$$\int_C f \cdot dR = \int_a^b f(R(t)) \cdot R'(t)dt$$

provided the integral in the right-hand side exists. Please see Figure 1.



Remark 0.1. Suppose $f = (f_1, f_2, f_3)$ and $R(t) = (x(t), y(t), z(t))$. Then the line integral $\int_C f \cdot dR$ is also written as

$$\int_C f_1 dx + f_2 dy + f_3 dz \qquad \text{or} \qquad \int_C f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz.$$

Example 0.2. Let $f = x^2 i + yj + (xz - y)k$. Compute the line integral $\int_C f \cdot dR$, along the curve C joining $(0, 0, 0)$ with $(1, 2, 4)$.
(i) C is the straight line joining theses points,
(ii) C is the curve given by $R(t) = (t^2, 2t, 4t^2)$.

The second FTC for line integral: We know that if $f : [a, b] \to \mathbb{R}$ is continuously differentiable on $[a, b]$, then $\int_a^b f(t)'dt = f(b) - f(a)$. Since f is continuous, $F(x) = \int_\alpha^x f(t)dt$ is differentiable and by FTC it follows that $F(x)' = f(x)$, where $\alpha \in [a, b]$. Hence $\int_a^b F'(x)dx = \int_a^b f(x)dx = \int_\alpha^b f(x)dx - \int_\alpha^a f(x)dx = F(b) - F(a)$. This says that the value of integral of continuously differentiable function depends only on end points and not on the points inside the interval.

We generalize the above second FTC to the line integral.

Theorem 0.3. *Let D be a solid domain in $\mathbb{R}^3$, and $f : D \to \mathbb{R}$ be continuously differentiable. Suppose A, B are two points in D. Let $C = \{R(t) : t \in [a, b]\}$ be a curve lying in D and joining the points A and B. If $R(t)$ is continuously differentiable on $[a, b]$, then*

$$\int_C \nabla f \cdot dR = f(B) - f(A).$$

Proof. Let $h(t) = f(R(t))$. Then by chain rule, we get $h'(t) = (f \circ R)'(t) = \nabla f(R(t)).R'(t)$. Hence

$$\int_C \nabla f \cdot dR = \int_a^b \nabla f(R(t)).R'(t)dt = \int_a^b h'(t)dt = h(b) - h(a) = f(B) - f(A).$$

□

Remark 0.4. Line integral of gradient of a function is independent of the choice of path joining the points A and B in the domain D.

Definition 0.5. Let $R : [a, b] \to \mathbb{R}^3$ be a continuous function that represents a curve C. The curve C is said to be

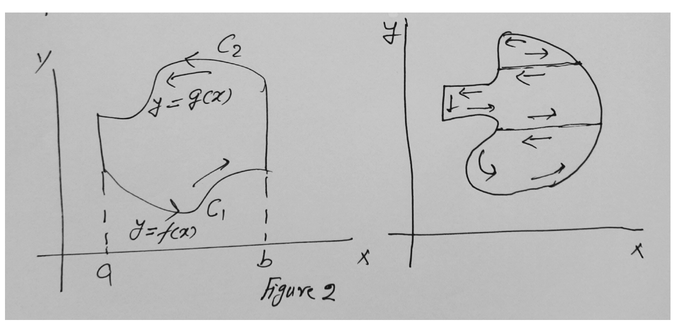
(i) **simple** if R is one-one on $(a, b]$.
(ii) **Closed** if $R(a) = R(b)$.
(iii) **Smooth** if R' exists and continuous.
(iv) **Piecewise smooth** if the interval $[a, b]$ can be partitioned into a finite number of subintervals such that R is smooth over each subinterval.

Theorem 0.6. *(Green's Theorem) Let C be a piecewise smooth simple closed curve in the xy-plane and let D denote the closed region enclosed by C. Suppose $M, N, \frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are real valued continuous functions in an open set containing D. Then*

$$\iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_C (Mi + Nj)dR = \oint_C Mdx + Ndy, \tag{0.1}$$

where the line integral is taken around C in the counterclockwise direction.

Since the identity (0.1) holds true for every choice of M and N (satisfying the assumption of Green's Theorem), by letting $M = 0$ and N arbitrary and vice-versa, the identity (0.1) is equivalent to two identities $\iint\limits_D \frac{\partial N}{\partial x} dxdy = \oint\limits_C Ndy$ and $-\iint\limits_D \frac{\partial M}{\partial y} dxdy = \oint\limits_C Mdy$. We shall present the proof of Green's Theorem for two special cases I and II as shown in Figure 2.



Proof. (*i*) Let $D = \{(x, y) : \ a \le x \le b \text{ and } f(x) \le y \le g(x)\}$, where f and g are continuous functions on $[a, b]$. Since $\frac{\partial M}{\partial y}$ is continuous, by Fubini's Theorem, the double integral

$$(0.2)\qquad -\iint\limits_{D} \frac{\partial \mathrm{M}}{\partial y} dxdy = \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial \mathrm{M}}{\partial y} dy \right] dx = \int_a^b M[x, f(x)]dx - \int_a^b M[x, g(x)]dx.$$

On the other hand, we can write

$$(0.3)\qquad \int_C M dx = \int_{C_1} M dx + \int_{C_2} M dx,$$

since the line integral along each of vertical segment is zero. Note that C_1 and C_2 can be represented by $r_1(t) = ti + f(t)j$ and $r_1(t) = ti + g(t)j$ respectively. Hence

$$(0.4)\qquad \int\limits_{C_1} M dx = \int_a^b M[t, f(t)]dt \text{ and } \int\limits_{C_2} M dx = -\int_a^b M[t, f(t)]dt.$$

Negative sign appeared in the second equation since the carve C_2 traverses in the reverse direction. Thus, from (0.2-0.4) we conclude that the identity (0.1) holds for the type I region. Similarly, we can obtain the result for the type II region. Further, we can obtain the result for any region which an be decomposed into finitely many regions of the above two types. □

Area expressed as a line integral: Let C be a simple (piecewise smooth) closed curve and D be the region enclosed by C. Let $M(x, y) = -\frac{y}{2}$ and $N(x, y) = \frac{x}{2}$. Then by Green's Theorem the area of D is

$$a(D) = \iint\limits_{D} dxdy = \int_D (N_x - M_y)\, dxdy = \int_a^b M dx + N dy = \frac{1}{2} \int_C -ydx + xdy.$$

Example 0.7. Note that the integral

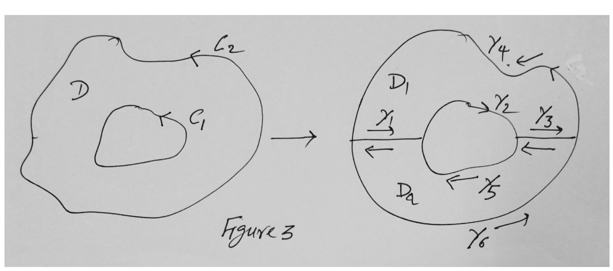
$$\int_C xy^2 dx + \left(x^2 y + x\right) dy = \iint\limits_{D} dxdy = \text{Area(D)}$$

(By Green's Theorem), where D is the region enclosed by C. Hence the integral is depending only on the region enclosed by C but not on its location.

Example 0.8. Find the area bounded by the ellipse $C = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$.
Consider the parametric form of $C = \{(a \cos t, b \sin t) :\ 0 \le t < 2\pi.\}$ Then the area is

$$\frac{1}{2} \int_C -ydx + xdy = \frac{1}{2} \int_0^{2\pi} -(b \sin t)(-a \sin t)dt + (a \cos t)(b \cos t)dt = \frac{1}{2} \int_0^{2\pi} abdt = ab\pi.$$

Example 0.9. Let C_1 and C_2 be two simple (piecewise smooth) closed curves as shown in Figure 3.



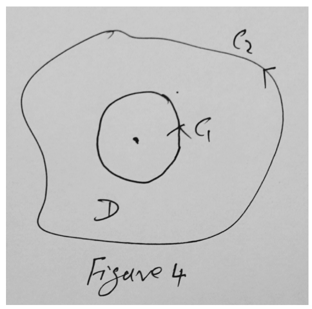
Consider the region D bounded by the curves C_1 and C_2. Note that $D = D_1 \cup D_2$ and D_1 is enclosed by the curves $\gamma_i;\ i = 1,2,3,4$ and D_2 is enclosed by curves $\gamma_j;\ j = 1,3,5,6$.

$$\iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint\limits_{D_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy + \iint\limits_{D_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \left(\int_{\gamma_1} \alpha + \int_{\gamma_2} \alpha + \int_{\gamma_3} \alpha + \int_{\gamma_4} \alpha\right) + \left(\int_{\gamma_6} \alpha - \int_{\gamma_3} \alpha + \int_{\gamma_5} \alpha - \int_{\gamma_1} \alpha\right) = \oint_{C_2} \alpha - \oint_{C_1} \alpha,$$

where $\alpha = Mdx + Ndy$.

Example 0.10. Let C_1 be unit circle and C_2 be any simple closed curve as shown in Figure 4.



Find $\int_{C_2} \frac{xdy - ydx}{x^2+y^2}$. Let D be the domain lies between C_1 and C_2. A simple calculation shows that $N_x - M_y = 0$ on D. By applying Green's Theorem for multiply-connected domain D, we get

$$\oint_{C_2} (Mdx + Ndy) - \oint_{C_1} (Mdx + Ndy) = \iint\limits_D (N_x - M_y)\, dxdy = 0.$$

Since $C_1 = \{(\cos t, \sin t) : 0 \le t \le 2\pi\}$, we get

$$\int_{C_1} \frac{-ydx + xdy}{x^2 + y^2} = \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\sin^2 t + \cos^2 t} dt = 2\pi.$$

Hence,

$$\oint_{C_2} (Mdx + Ndy) = 2\pi.$$

Exactness of the line integral. Let Q be a cube in $\mathbb{R}^3$. Suppose C is a curve in Q which is parameterized by $R(t) = (x(t), y(t), z(t))$, where $R : [a,b] \to \mathbb{R}^3$ is continuously differentiable. Now, does there exist a function $F : Q \subset \mathbb{R}^3 \to \mathbb{R}$ such that $\int\limits_C f \cdot dR = \int\limits_C dF$ for every curve C in Q? Suppose there exists F such that $\int\limits_C f \cdot dR = \int\limits_C dF$ for every curve C in Q. Then by Theorem 0.3 (second FTC for line integral), it follows that

$$\int_C f \cdot dR = F(R(b)) - F(R(a)) = F(B) - F(A) = \int_C \nabla F \cdot dR.$$

That is,

$$\int_C (f - \nabla F) \cdot dR = 0$$

for all curves C in Q. It is easy to see that $f = \nabla F$ on Q.

Remark 0.11. Note that it is difficult to prove that $f = \nabla F$ on a general domain D. However, the following exercise can be done with small effort.

Exercise 0.12. Let $D = \{(x,y) : \ x^2 + y^2 < 1\}$. If $f : D \to R^2$ is a continuously differentiable function such that $\int_\Gamma f \cdot dR = 0$ for every curve Γ in D, then f constant.

Example 0.13. Show that the line integral

$$\int_C 2x \sin y \ dx + (x^2 \cos y - 3y^2) dy$$

is path independent joining the points $(-1, 0)$ and $(5, 1)$.

Curl and Divergence

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$.

Definition 0.14. (Curl of F) The curl of F is another vector field denoted by curl F and defined by the vector

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times f,$$

where $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$.

Definition 0.15. (Divergence of F) The divergence of F is a scalar valued function denoted by div F and is defined by $\text{div}\, F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. We can rewrite the div F as div $F = \nabla \cdot F$.

Now, we recall Green's Theorem to get a motivation for Stoke's Theorem. Let C be the piece-wise smooth curve which encloses the domain D in $\mathbb{R}^2$. Let $F : D \to \mathbb{R}^2$ be a vector field in the plane given by $F(x,y) = M(x,y)i + N(x,y)j + 0k$. By Green's Theorem

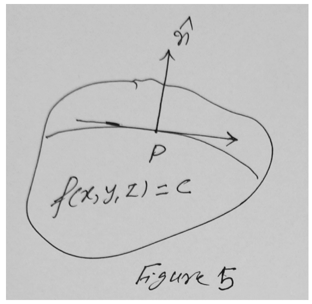
$$\iint_D (N_x - M_y)\, dxdy = \oint_C M dx + N dy,$$

where $C = \{R(t) : t \in [a,b]\}$. The above identity can be represented as

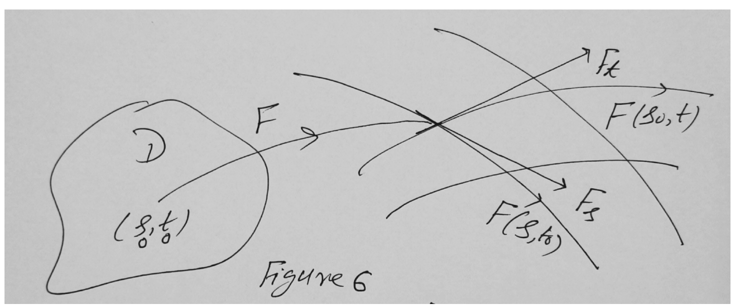
$$\iint_D \text{curl } F \cdot k \ dxdy = \oint_C F \cdot dR, \tag{0.5}$$

where $\text{curl } F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) k$. Stoke's Theorem is a generalization of the identic (0.5) in $\mathbb{R}^3$. Before we make a formal statement for Stoke's Theorem, we discuss unit normal vector on some special surfaces.

(i) Suppose the surface S is given by $f(x,y,z) = c$, where f is differentiable function on some domain D in $\mathbb{R}^3$. Please see Figure 5.



Consider a smooth curve C given by $R : [a, b] \to \mathbb{R}^3$ which lies on the surface S and passes through a point P on S. Then $f(R(t)) = c$. By the chain rule, we get $f'(R(t)).R'(t) = 0$. That is, $\nabla f(R(t)).R'(t) = 0$. Since $R'(t)$ is the tangent vector at point P, the vector $\nabla f(R(t))$ is the normal vector at P. Hence the unit normal vector $\hat{n}$ is given by $\hat{n} = \frac{\nabla f}{\|\nabla f\|}$. Note that $\hat{n} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$. If $\hat{n}$ is continuous and never vanishes on D, then the surface S is called **orientable.**
(ii) Let D be a domain in $\mathbb{R}^2$. Let $F : D \to \mathbb{R}^3$ given by $F(s, t) = x(s, t)i+y(s, y)j+z(s, t)k$ is a parametrization of surface S, where F is smooth (continuously differentiable). Let $P = F(s_o, t_o)$ be a point on the surface S. Then $F(s, t_o)$ and $F(s_o, t)$ are curves on S passing through P as shown in Figure 6.



Recall that the fundamental product $F_s \times F_t$ is the normal to the surface S at P. Hence unit normal vector to the surface S, in this case, is given by $\hat{n} = \frac{F_s \times F_t}{\|F_s \times F_t\|}$. Note that $\hat{n} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$.
(iii) If the surface S is given by the graph of smooth function $f : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. That is, $F(x, y) = xi + yj + f(x, y)k$. Then unit normal vector is given by

$$\hat{n} = \frac{F_x \times F_y}{\|F_x \times F_y\|} = \frac{-f_x i - f_y j + k}{\| - f_x i - f_y j + k\|} = \frac{-f_x i - f_y j + k}{\sqrt{1 + f_x^2 + f_y^2}}.$$

Definition 0.16. A surface S is called orientable if unit normal vector to the surface S is continuous and never vanishes.

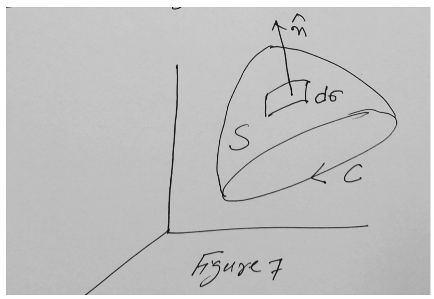
Hence **orientable surface** is a two-sided surface. Möbius strip is not an orientable surface.

Theorem 0.17. (Stokes' Theorem) Let S be a piecewise smooth orientable surface and C be the piecewise smooth boundary of S. Let $F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$ be a vector field such that P, Q and R are continuously differentiable on an open set containing S. If $\hat{n}$ is a unit normal vector to S, then

$$\iint\limits_S \text{curl } F \cdot \hat{n}\, d\sigma = \oint_C F \cdot dR, \tag{0.6}$$

where the line integral is evaluated around C in the direction of the orientation of C with respect to $\hat{n}$.

Please see Figure 7.



(i) Note that the value of surface integral in (0.6) depends only on the boundary C and not to the shape of the surface S.
(ii) If S is a plane surface, then identity (0.6) reduces to the identity (0.5). Thus, Stoke's Theorem can be considered as a direct extension of Green's Theorem.
(iii) For the closed smooth surface like sphere and donut, there is no boundary and in this case $\iint\limits_S \text{curl } F \cdot \hat{n}\, d\sigma = 0$.
(iv) Stoke's Theorem can be extended to a smooth surface whose boundary contains more than one simple smooth closed curve.

Remark 0.18. If a surface S is given by the graph of smooth function f defined on the domain $D \subset \mathbb{R}^2$, then

$$\oint_C F \cdot dR = \iint\limits_D (-f_x i - f_y j + k) \cdot \text{curl } F\ dxdy.$$

Example 0.19. Let S be the part of the cylinder $z = 1 - x^2,\ 0 \le x \le 1,\ -2 \le y \le 2$. Let C be the boundary of the surface S and $F(x, y, z) = yi + yj + k$. Use Stoke's Theorem to find the line integral $\int\limits_C F \cdot dR$.

Here curl $F = -k$. Let $z = f(x, y) = 1 - x^2$. The unit normal to the surface S will be given by

$$\hat{n} = \frac{-f_x i - f_y j + k}{\sqrt{1 + f_x + f_y^2}} = \frac{2xi + k}{\sqrt{1 + 4x^2}}.$$

Surface element $d\sigma(x, y) = \sqrt{1 + f_x + f_y^2}\ dxdy$. Please refer to Figure 8.

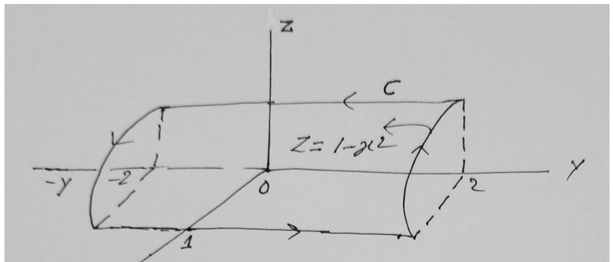


Figure 8

By stoke's Theorem as mentioned in Remark 0.18,

$$\oint_C F\cdot dR = \iint_D (-f_x i - f_y j + k)\cdot \text{curl } F\ dxdy = \int_{y=-2}^{2}\int_{x=0}^{1}(-1)dxdy = -4.$$

Once again let us look at the Green's Theorem in the plane. Let $F(x,y) = M(x,y)i + N(x,y)j$ be smooth on the domain $D \subset \mathbb{R}^2$, where D is enclosed by the simple and smooth curve $C = \{R(t):\ t\in[a,b]\}$. Then $R'(t) = x'(t)i + y'(t)j$ is the tangent vector to the curve. Hence $n = y'(t)i - x'(t)j$ is a normal vector to the curve C. By Green's Theorem

$$\oint_C (F\cdot n)dt = \oint_C Mdy - Ndx = \iint_D \left(\frac{\partial M}{\partial x} - \left(-\frac{\partial N}{\partial y}\right)\right)dxdy = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dxdy.$$

Hence

$$\iint_D \text{div } F\ dxdy = \oint_C (F\cdot n)ds. \tag{0.7}$$

The generalization of the identity (0.7) to $\mathbb{R}^3$ is known as the divergence theorem.

Theorem 0.20. (Divergence Theorem) Let D be a solid domain in $\mathbb{R}^3$ bounded by piecewise smooth and orientable surface S. Let $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$ be vector filed which is continuously differentiable on an open set that contains D. Let $\hat{n}$ be the unit outward normal to the surface S. Then

$$\iiint_D \text{div}\, F\ dV = \iint_S F\cdot\hat{n}\ d\sigma.$$

Refer to Figure 9.

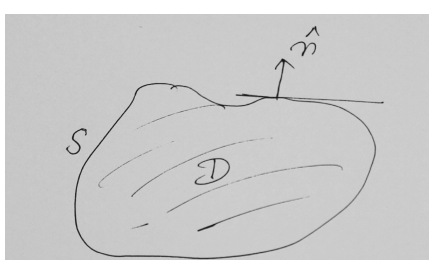


Figure 9

Example 0.21. Let $F(x,y,z) = (x+y)i + z^2 j + x^2 k$. Let $\hat{n}$ be the unit outward normal to the hemisphere $S = \{(x,y,z) : x^2+y^2+z^2 = 1 \text{ and } z > 0\}$. Find the surface integral $\iint\limits_S F \cdot \hat{n}\, d\sigma$ using divergence theorem.

Let $F(x,y,z) = (x+y, z^2, x^2)$. Then div $F = 1$. Note that S is not a closed surface. Let $S_1 = \{(x,y) :\ x^2+y^2 \leq 1\}$. Then $S \cup S_1$ is a closed surface and we can apply divergence theorem for it. Please refer to Figure 10.

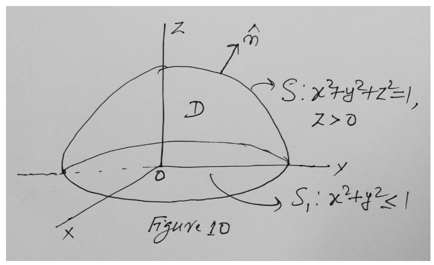


Figure 10

By divergence theorem,

$$\iint_S F \cdot \hat{n}\, d\sigma + \iint_{S_1} F \cdot \hat{n_1}\, d\sigma_1 = \iiint\limits_D \text{div } F\, dV = \frac{2\pi}{3}.$$

Here

$$\iint_{S_1} F \cdot \hat{n_1}\, d\sigma_1 = \iint_{S_1} (x+y, z^2, x^2).(-k) dxdy = -\iint_{x^2+y^2 \leq 1} x^2 dxdy.$$