MA15010H: Multi-variable Calculus

(Practice problem set 1: Limits and continuity)

July - November, 2025

- 1. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then show that
 - (a) $\|\mathbf{x} \mathbf{y}\| \le \|\mathbf{x} \mathbf{y}\|$.
 - (b) $\|\mathbf{x} + \mathbf{y}\| \|\mathbf{x} \mathbf{y}\| \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.
 - (c) $\|\mathbf{x}\| \le \max\{\|\mathbf{x} + \mathbf{y}\|, \|\mathbf{x} \mathbf{y}\|\}.$
 - (d) $\|\mathbf{x} + \alpha \mathbf{y}\| \ge \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ iff $\mathbf{x} \cdot \mathbf{y} = 0$.
- 2. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $\alpha > 0$. Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \alpha ||\mathbf{x}||^2 + \frac{1}{4\alpha} ||\mathbf{y}||^2$.
- 3. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Show that $\|\mathbf{x}\| \|\mathbf{y}\| = \|\mathbf{x} \mathbf{y}\|$ iff $\alpha \mathbf{x} = \beta \mathbf{y}$ for some $\alpha, \beta \geq 0$ with $(\alpha, \beta) \neq (0, 0)$.
- 4. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and r > 0 such that $\mathbf{y} \cdot \mathbf{z} = 0$ for all $\mathbf{z} \in B_r(\mathbf{x})$. Show that $\mathbf{y} = 0$.
- 5. If $x_0 \in \mathbb{R}^m$ and r > 0, then determine $\sup\{\|\mathbf{x} \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in B_r(x_0)\}$ with justification.
- 6. Let $S \subseteq \mathbb{R}^m$ such that $S \subseteq B_r(x_0)$ for some $x_0 \in \mathbb{R}^m$ and for some r > 0. Show that S is a bounded set.
- 7. Let $\alpha \in (0,1)$ and let $\mathbf{x}_n = (n^{3\alpha}n, \frac{1}{n}[n\alpha])$ for all $n \in \mathbb{N}$ (For each $x \in \mathbb{R}$, [x] denotes the greatest integer not exceeding x). Examine whether the sequence (\mathbf{x}_n) converges in \mathbb{R}^2 . Also, find $\lim_{n\to\infty} \mathbf{x}_n$ if the sequence (\mathbf{x}_n) converges in \mathbb{R}^2 .
- 8. Let (\mathbf{x}_n) be a sequence in \mathbb{R}^m such that the series $\sum_{n=1}^{\infty} n^2 ||\mathbf{x}_n||^2$ is convergent. Show that the series $\sum_{n=1}^{\infty} ||\mathbf{x}_n||$ is convergent.
- 9. Let (\mathbf{x}_n) and (\mathbf{y}_n) be sequences in \mathbb{R}^m such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y}_n \to \mathbf{y} \in \mathbb{R}^m$. Show that $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$ and $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$.
- 10. Let $\mathbf{x} \in \mathbb{R}^m$ and let (\mathbf{x}_n) be a sequence in \mathbb{R}^m such that $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$ and $\mathbf{x}_n \cdot \mathbf{x}_n \to \mathbf{x} \cdot \mathbf{x}$. Show that (\mathbf{x}_n) is convergent.
- 11. State TRUE or FALSE with justification for each of the following statements:
 - (a) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} \neq \mathbf{y}$ and $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$, then it is necessary that $\|\mathbf{x} + \mathbf{y}\| < 2$.
 - (b) If (\mathbf{x}_n) is a sequence in \mathbb{R}^m such that for each $\mathbf{x} \in \mathbb{R}^m$, $\lim_{n \to \infty} \mathbf{x}_n \cdot \mathbf{x}$ exists (in \mathbb{R}), then $\lim_{n \to \infty} ||\mathbf{x}_n||^2$ must exist (in \mathbb{R}).
 - (c) There exists an unbounded sequence (x_n) of distinct real numbers such that the sequence $(x_n, \cos x_n)$ in \mathbb{R}^2 has a convergent subsequence.

- 12. Let $S = \{(x,y) \in \mathbb{R}^2 : x \neq y\}$ and let $f: S \to \mathbb{R}$ be defined by $f(x,y) = \frac{x+y}{x-y}$ for all $(x,y) \in S$. Show by using the definition of continuity that f is continuous at (1,2).
- 13. If $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(x,y) = x^2 + y^2$ for all $x \in \mathbb{Q}$ and for all $y \in \mathbb{R} \setminus \mathbb{Q}$, then determine $f(\sqrt{2},2)$.
- 14. Examine the continuity of $f: \mathbb{R}^2 \to \mathbb{R}$ at (0,0), where for all $(x,y) \in \mathbb{R}^2$,

(a)
$$f(x,y) = \begin{cases} xy & \text{if } xy \ge 0, \\ -xy & \text{if } xy < 0. \end{cases}$$

(b) $f(x,y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x,y) \ne (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
(c) $f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$

15. Determine all the points of \mathbb{R}^2 where $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous, if for all $(x, y) \in \mathbb{R}^2$,

label=(a)
$$f(x,y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

lbbel=(b) $f(x,y) = \begin{cases} xy & \text{if } xy \in \mathbb{Q}, \\ -xy & \text{if } xy \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

16. Let α, β be positive real numbers and let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{|x|^{\alpha}|y|^{\beta}}{x^2 + x^2y^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is continuous iff $\alpha + \beta > 2$.

- 17. Let S be a nonempty subset of \mathbb{R}^m and let $f_j: S \to \mathbb{R}$ for each $j \in \{1, \ldots, k\}$. If $f(x) = (f_1(x), \ldots, f_k(x))$ for all $x \in S$, then show that $f: S \to \mathbb{R}^k$ is continuous at $x_0 \in S$ iff f_j is continuous at x_0 for each $j \in \{1, \ldots, k\}$.
- 18. Examine the continuity of $f: \mathbb{R}^2 \to \mathbb{R}^2$ at (0,0), where for all $(x,y) \in \mathbb{R}^2$,

$$f(x,y) = \begin{cases} \left(\frac{x^3}{x^2 + y^2}, \sin(x^2 + y^2)\right) & \text{if } (x,y) \neq (0,0), \\ (0,0) & \text{if } (x,y) = (0,0). \end{cases}$$

19. If $f, g: S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ are continuous at $x_0 \in S$ and if $\varphi(x) = f(x) \cdot g(x)$ for all $x \in S$, then show that $\varphi: S \to \mathbb{R}$ is continuous at x_0 .

- 20. Let $f: S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous at $x_0 \in S^0$ and let $f(x_0) \neq 0$. Show that there exists r > 0 such that $f(x) \neq 0$ for all $x \in B_r(x_0)$.
- 21. Let S be an open subset of \mathbb{R}^m and let $f: S \to \mathbb{R}^k$ and $g: S \to \mathbb{R}^k$ be continuous at $x_0 \in S$. If for each $\epsilon > 0$, there exist $x, y \in B_{\epsilon}(x_0)$ such that f(x) = g(y), then show that $f(x_0) = g(x_0)$.
- 22. If $S = \{(x, y) \in \mathbb{R}^2 : x + y \ge 2\}$, then determine (with justification) S^0 .
- 23. If $S = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m = 1\}$, then determine (with justification) S^0 .
- 24. If $\mathbf{x} \in \mathbb{R}^m$ and r > 0, then determine (with justification) all the interior points of $B_r[\mathbf{x}]$.
- 25. Examine whether $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ is an open set in \mathbb{R}^2 .