MA15010H: Multi-variable Calculus

(Practice problem set 2: Hint/Model solution) July - November, 2025

1. Examine whether the set $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ is (a) open (b) closed in \mathbb{R}^2 .

Solution: We have already shown in Ex.25 of Practice Problem Set - 1 that $S = \{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ is an open set in \mathbb{R}^2 .

Again, since $\left(\frac{1}{2n}, \frac{1}{n}\right) \in S$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{2n}, \frac{1}{n}\right) \to (0, 0) \notin S$, S is not a closed set in \mathbb{R}^2 .

2. Examine whether the set $\{(x,x):x\in\mathbb{R}\}$ is (a) open (b) closed in \mathbb{R}^2 .

Solution: We have $(0,0) \in S = \{(x,x) : x \in \mathbb{R}\}$. If possible, let $(0,0) \in S^0$. Then there exists r > 0 such that $B_r((0,0)) \subset S$. Since $(\frac{r}{2},0) \in B_r((0,0))$ but $(\frac{r}{2},0) \notin S$, we get a contradiction. Hence $(0,0) \notin S^0$. Therefore S is not an open set in \mathbb{R}^2 .

Again, let $((x_n, x_n))$ be any sequence in S such that $(x_n, x_n) \to (x, y)$ in \mathbb{R}^2 . Then $x_n \to x$ and $x_n \to y$. Hence x = y and so $(x, y) \in S$. Therefore S is a closed set in \mathbb{R}^2 .

3. Examine whether the set $\{(x,y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$ is (a) open (b) closed in \mathbb{R}^2 .

Solution: We have $(0,0) \in S = \{(x,y) : y \in \mathbb{Z}\}$. If possible, let $(0,0) \in S^0$. Then there exists r > 0 such that $B_r((0,0)) \subset S$. If $s = \min\{\frac{1}{2}, \frac{r}{2}\}$, then $(0,s) \in B_r((0,0))$ but $(0,s) \notin S$. Thus we get a contradiction. Hence $(0,0) \notin S^0$ and therefore S is not an open set in \mathbb{R}^2 .

Again, let $((x_n, y_n))$ be any sequence in S such that $(x_n, y_n) \to (x, y)$ in \mathbb{R}^2 . Then $y_n \to y$. There exists $N \in \mathbb{N}$ such that $|y_n - y| < \frac{1}{2}$ for all $n \geq N$ and hence $|y_n - y_0| \leq |y_n - y| + |y - y_0| < 1$ for all $n \geq n_0$. Since $y_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, we get $y_n = y_0$ for all $n \geq n_0$ and so $y_n \to y_0$. Consequently $y = y_0 \in \mathbb{Z}$ and so $(x, y) \in S$. Therefore S is a closed set in \mathbb{R}^2 .

4. Examine whether the set $(0,1) \times \{0\}$ is (a) open (b) closed in \mathbb{R}^2 .

Solution: We have $(\frac{1}{2},0) \in (0,1) \times \{0\}$. If possible, let $(\frac{1}{2},0) \in ((0,1) \times \{0\})^0$. Then there exists r > 0 such that $B_r\left((\frac{1}{2},0)\right) \subset (0,1) \times \{0\}$. Since $(\frac{1}{2},\frac{r}{2}) \in B_r\left((\frac{1}{2},0)\right)$ but $(\frac{1}{2},\frac{r}{2}) \notin (0,1) \times \{0\}$, we get a contradiction. Hence $(\frac{1}{2},0) \notin ((0,1) \times \{0\})^0$. Therefore $(0,1) \times \{0\}$ is not an open set in \mathbb{R}^2 .

Again, since $\left(\frac{1}{n+1},0\right) \in (0,1) \times \{0\}$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{n+1},0\right) \to (0,0) \notin (0,1) \times \{0\}$, $(0,1) \times \{0\}$ is not a closed set in \mathbb{R}^2 .

5. If $f: \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ is an open set in \mathbb{R}^m .

Solution: Let (\mathbf{x}_n) be any sequence in $\mathbb{R}^m \setminus S$, where $S = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at \mathbf{x} , $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in \mathbb{R}^m \setminus S$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) \leq 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) \leq 0$. Thus $\mathbf{x} \in \mathbb{R}^m \setminus S$ and therefore $\mathbb{R}^m \setminus S$ is a closed set in \mathbb{R}^m . Consequently S is an open set in \mathbb{R}^m .

6. If $f: \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \geq 0\}$ and $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0\}$ are closed sets in \mathbb{R}^m .

Solution: Let (\mathbf{x}_n) be any sequence in $S_1 = {\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \ge 0}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at \mathbf{x} , $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in S_1$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) \ge 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) \ge 0$. Thus $\mathbf{x} \in S_1$ and therefore S_1 is a closed set in \mathbb{R}^m .

Again, let (\mathbf{x}_n) be any sequence in $S_2 = {\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at \mathbf{x} , $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in S_2$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) = 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) = 0$. Thus $\mathbf{x} \in S_2$ and therefore S_2 is a closed set in \mathbb{R}^m .

7. Using Ex.2 in the Practice Problem Set - 2, show that $\{(x,y,z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\}$ is an open set in \mathbb{R}^3 and $\{(x,y,z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\}$ is a closed set in \mathbb{R}^3 .

Solution: If $f(x,y,z) = 3|y| - x^2 - 2z$ and $g(x,y,z) = \sin(xyz) - |xy|$ for all $(x,y,z) \in \mathbb{R}^3$, then we know that both $f: \mathbb{R}^3 \to \mathbb{R}$ and $g: \mathbb{R}^3 \to \mathbb{R}$ are continuous. Hence by Ex.2(a) of Practice Problem Set - 2, $\{(x,y,z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\} = \{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) > 0\}$ is an open set in \mathbb{R}^3 and by Ex.2(b) of Practice Problem Set - 2, $\{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = 0\}$ is a closed set in \mathbb{R}^3 .

8. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$$

Show that f is continuous.

Solution: If $\varphi(x,y) = xy$ and $\psi(x,y) = \sin(xy)$ for all $(x,y) \in \mathbb{R}^2$, then we know that $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $\varphi(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$. Hence it follows that f is continuous at each point $(x,y) \in \mathbb{R}^2$ for which $xy \neq 0$.

Let $(x,y) \in \mathbb{R}^2$ such that xy = 0 and let $((x_n, y_n))$ be any sequence in \mathbb{R}^2 such that $(x_n, y_n) \to (x, y)$. Then $x_n \to x, y_n \to y$ and so $x_n y_n \to xy = 0$. Now $f(x_n, y_n) = \frac{\sin(x_n y_n)}{x_n y_n}$ if $x_n y_n \neq 0$ and $f(x_n, y_n) = 1$ if $x_n y_n = 0$. Since $\lim_{t\to 0} \frac{\sin t}{t} = 1$, it follows that $f(x_n, y_n) \to 1 = f(x, y)$ and consequently f is continuous at (x, y). Therefore f is continuous.

9. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x,y) = e^{-\frac{x^2 - 2xy + y^2}{|x-y|}}$ for all $(x,y) \in \mathbb{R}^2$ with $x \neq y$. If $x \in \mathbb{R}$, then find f(x,x) such that f is continuous on \mathbb{R}^2 .

Solution: Since $x^2-2xy+y^2=|x-y|^2$ for all $x,y\in\mathbb{R}$, we find that $f(x,y)=e^{-|x-y|}$ for all $(x,y)\in\mathbb{R}^2$ with $x\neq y$. If $x\in\mathbb{R}$, then $(x+\frac{1}{n},x)\to(x,x)$ and for f to be continuous at (x,x), we must have $f(x,x)=\lim_{n\to\infty}f(x+\frac{1}{n},x)=\lim_{n\to\infty}e^{-\frac{1}{n}}=1$. So, let f(x,x)=1 for all $x\in\mathbb{R}$.

If g(x,y) = -|x-y| for all $(x,y) \in \mathbb{R}^2$ and $\varphi(t) = e^t$ for all $t \in \mathbb{R}$, then $f(x,y) = \varphi(g(x,y))$ for all $(x,y) \in \mathbb{R}^2$. Since we know that $g: \mathbb{R}^2 \to \mathbb{R}$ and $\varphi: \mathbb{R} \to \mathbb{R}$ are continuous, hence $f = \varphi \circ g$ is also continuous.

- 10. Let $f: S \subset \mathbb{R}^m \to \mathbb{R}^k$ be continuous and let $g: \mathbb{R}^m \to \mathbb{R}^k$ be such that g(x) = f(x) for all $x \in S$.
 - (a) Show that g need not be continuous on S.
 - (b) If S is an open set in \mathbb{R}^m , then show that g is continuous on S.

Solution:

(a) Let f(x,y) = 1 for all $(x,y) \in S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ and

$$g(x,y) = \begin{cases} 1 & \text{if } (x,y) \in S, \\ 2 & \text{if } (x,y) \in \mathbb{R}^2 \backslash S. \end{cases}$$

Then $f: S \to \mathbb{R}$ is continuous (as a constant function) and f(x,y) = g(x,y) for all $(x,y) \in S$. However, g is not continuous at $(1,0) \in S$, since $(1+\frac{1}{n},0) \to (1,0)$ but

$$g(1+\frac{1}{n},0)=2\to 2\neq 1=g(1,0).$$

- (b) Let $x_0 \in S$ and $\varepsilon > 0$. Since S is an open set in \mathbb{R}^m , there exists r > 0 such that $B_r(x_0) \subset S$. Since f is continuous at x_0 , there exists s > 0 such that $||f(x) f(x_0)|| < \varepsilon$ for all $x \in S \cap B_s(x_0)$. If $\delta = \min\{r, s\} > 0$, then $B_\delta(x_0) \subset B_r(x_0) \subset S$ and $B_\delta(x_0) \subset B_s(x_0)$. Hence for all $x \in B_\delta(x_0)$, we have g(x) = f(x) and $||g(x) g(x_0)|| < \varepsilon$. Therefore g is continuous at x_0 . Since $x_0 \in S$ is arbitrary, g is continuous on S.
- 11. Let $S_1 = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 4\}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 9\}$. Does there exist a continuous function from S_1 onto S_2 ? Justify.

Solution: Let u = (1,0), v = (0,1) and let $f(x) = v + \frac{3}{2}(x-u) = \left(\frac{3x}{2} - \frac{3}{2}, 1 + \frac{3y}{2}\right)$ for all $x = (x,y) \in S_1$. If $x \in S_1$, then $||f(x) - v|| = \frac{3}{2}||x - u|| < 3$ and so $f(x) \in S_2$. Thus f maps S_1 to S_2 and clearly f is continuous (since both the component functions of f are continuous).

Again, if $y \in S_2$, then $x = u + \frac{2}{3}(y - v) \in \mathbb{R}^2$ and $||x - u|| = \frac{2}{3}||y - v|| < 2$, i.e. $x \in S_1$, and also f(x) = y. Thus $f: S_1 \to S_2$ is onto. Therefore there exists a continuous function from S_1 onto S_2 .

12. If $S = \{ \mathbf{x} \in \mathbb{R}^m : ||\mathbf{x}|| < 1 \}$, then does there exist a non-constant continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(\mathbf{x}) = 5$ for all $\mathbf{x} \in S$? Justify.

Solution: There exists such a function as is shown by the following example. Let

$$f(\mathbf{x}) = \begin{cases} 5 & \text{if } \mathbf{x} \in S, \\ 5\|\mathbf{x}\| & \text{if } \mathbf{x} \in \mathbb{R}^m \setminus S. \end{cases}$$

If (\mathbf{x}_n) is any sequence in \mathbb{R}^m such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$, then using Ex.1(a) of Practice Problem Set - 1, we get $||\mathbf{x}_n|| - ||\mathbf{x}||| \le ||\mathbf{x}_n - \mathbf{x}|| \to 0$ and hence $||\mathbf{x}_n|| \to ||\mathbf{x}||$. It follows that $f : \mathbb{R}^m \to \mathbb{R}$ is continuous. Clearly f is a non-constant function and $f(\mathbf{x}) = 5$ for all $\mathbf{x} \in S$.

13. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ such that $x \neq y$. Find a continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(\mathbf{x}) = 1, f(\mathbf{y}) = 0$ and $0 \leq f(\mathbf{z}) \leq 1$ for all $\mathbf{z} \in \mathbb{R}^m$.

Solution: Let $f(\mathbf{z}) = \frac{\|\mathbf{z} - \mathbf{y}\|}{\|\mathbf{z} - \mathbf{x}\| + \|\mathbf{z} - \mathbf{y}\|}$ for all $\mathbf{z} \in \mathbb{R}^m$. If (\mathbf{z}_n) is any sequence in \mathbb{R}^m such that $\mathbf{z}_n \to \mathbf{z} \in \mathbb{R}^m$, then using Ex.1(a) of Practice Problem Set - 1, we find that $\|\mathbf{z}_n - \mathbf{x}\| \to \|\mathbf{z} - \mathbf{x}\|$, $\|\mathbf{z}_n - \mathbf{y}\| \to \|\mathbf{z} - \mathbf{y}\|$. Also, $\|\mathbf{v} - \mathbf{x}\| + \|\mathbf{v} - \mathbf{y}\| \neq 0$ for all $\mathbf{v} \in \mathbb{R}^m$. Hence it follows that $f(\mathbf{z}_n) \to f(\mathbf{z})$ and consequently $f: \mathbb{R}^m \to \mathbb{R}$ is continuous. Clearly $f(\mathbf{x}) = 1$, $f(\mathbf{y}) = 0$ and $0 \leq f(\mathbf{z}) \leq 1$ for all $\mathbf{z} \in \mathbb{R}^m$.

14. Let $f: \mathbb{R}^m \to \mathbb{R}$ be continuous such that $\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = 1$. Show that f is bounded on \mathbb{R}^m .

Solution: Since $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = 1$, there exists r > 0 such that $|f(\mathbf{x}) - 1| < 1$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| > r$. Hence $|f(\mathbf{x})| = |f(\mathbf{x}) - 1 + 1| \le |f(\mathbf{x}) - 1| + 1 < 2$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| > r$. Again, since $S = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| \le r\}$ is a closed and bounded subset of \mathbb{R}^m and since $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, f(S) is a bounded subset of \mathbb{R} . Hence there exists K > 0 such that $|f(\mathbf{x})| \le K$ for all $\mathbf{x} \in S$. If $M = \max\{2, K\}$, then M > 0 and $|f(\mathbf{x})| \le M$ for all $\mathbf{x} \in \mathbb{R}^m$. Consequently f is bounded on \mathbb{R}^m .

15. State TRUE or FALSE with justification: There exists r > 0 such that $\sin(xy) < \cos(xy)$ for all $x, y \in [-r, r]$.

Solution: If $f(x,y) = \sin(xy) - \cos(xy)$ for all $(x,y) \in \mathbb{R}^2$, then we know that $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (0,0) and f(0,0) = -1 < 0. Hence there exists $\delta > 0$ such that f(x,y) < 0, i.e. $\sin(xy) < \cos(xy)$ for all $(x,y) \in B_{\delta}((0,0))$. If $r = \frac{\delta}{2} > 0$, then $[-r,r] \times [-r,r] \subseteq B_{\delta}((0,0))$ and hence for all $x,y \in [-r,r]$, we have $(x,y) \in B_{\delta}((0,0))$ and consequently $\sin(xy) < \cos(xy)$. Therefore the given statement is TRUE.

16. State TRUE or FALSE with justification: There exists a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\cos n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.

Solution: Since $(\cos n)$ is a bounded sequence in \mathbb{R} , by Bolzano-Weierstrass theorem in \mathbb{R} , there exists a strictly increasing sequence (n_k) in \mathbb{N} and $\alpha \in \mathbb{R}$ such that $\cos n_k \to \alpha$. If $f: \mathbb{R} \to \mathbb{R}^2$ is continuous, then $\left(n_k, \frac{1}{n_k}\right) = f(\cos n_k) \to f(\alpha)$ in \mathbb{R}^2 and consequently the sequence $\left(n_k, \frac{1}{n_k}\right)$ converges in \mathbb{R}^2 , which is not true, since (n_k) is unbounded. Hence it follows that no continuous function $f: \mathbb{R} \to \mathbb{R}^2$ can exist satisfying $f(\cos n) = \left(n, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$. Therefore the given statement is FALSE.

17. State TRUE or FALSE with justification: There exists a continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ onto \mathbb{R}^2 .

Solution: We know that $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} = B_1[(0,0)]$ is a closed and bounded set in \mathbb{R}^2 and \mathbb{R}^2 is not bounded. Hence there cannot exist any continuous function from $B_1[(0,0)]$ onto \mathbb{R}^2 .

18. State TRUE or FALSE with justification: There exists a one-one continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ onto \mathbb{R}^2 .

Solution: Let $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and let $f(\mathbf{x}) = \frac{1}{1-\|\mathbf{x}\|}\mathbf{x} = \left(\frac{x}{1-\sqrt{x^2+y^2}}, \frac{y}{1-\sqrt{x^2+y^2}}\right)$ for all $\mathbf{x} = (x,y) \in S$. If $\mathbf{x} \in S$ and (\mathbf{x}_n) is any sequence in S such that $\mathbf{x}_n \to \mathbf{x}$, then using Ex.1(a) of Practice Problem Set - 1, we get $\|\mathbf{x}_n\| - \|\mathbf{x}\| \le \|\mathbf{x}_n - \mathbf{x}\| \to 0$ and so

 $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$. Hence $1 - \|\mathbf{x}_n\| \to 1 - \|\mathbf{x}\|$ and since $1 - \|\mathbf{x}\| \neq 0$ and $1 - \|\mathbf{x}_n\| \neq 0$ for all $n \in \mathbb{N}$, it follows that $f(\mathbf{x}_n) \to f(\mathbf{x})$. Therefore $f: S \to \mathbb{R}^2$ is continuous at \mathbf{x} and since $\mathbf{x} \in S$ is arbitrary, f is continuous.

Let $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that $f(\mathbf{x}_1) = f(\mathbf{x}_2)$. Then $||f(\mathbf{x}_1)|| = ||f(\mathbf{x}_2)||$, i.e. $\frac{||\mathbf{x}_1||}{1-||\mathbf{x}_1||} = \frac{||\mathbf{x}_2||}{1-||\mathbf{x}_2||}$, which gives $||\mathbf{x}_1|| = ||\mathbf{x}_2||$. Consequently from $\frac{1}{1-||\mathbf{x}_1||}\mathbf{x}_1 = \frac{1}{1-||\mathbf{x}_2||}\mathbf{x}_2$, we get $\mathbf{x}_1 = \mathbf{x}_2$. Hence f is one-one.

Again, if $\mathbf{y} \in \mathbb{R}^2$, then taking $\mathbf{x} = \frac{1}{1+|\mathbf{y}|}\mathbf{y}$, we find that $||\mathbf{x}|| < 1$, i.e. $\mathbf{x} \in S$ and $f(\mathbf{x}) = \mathbf{y}$. Hence f is onto.

Thus $f: S \to \mathbb{R}^2$ is the required function and therefore the given statement is TRUE.

19. If $f: \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, then does there exist a sequence $((x_n, y_n))$ in \mathbb{R}^2 such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$? Justify.

Solution: If possible, let there exist a sequence $((x_n, y_n))$ in \mathbb{R}^2 such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$. Then $||(x_n, y_n)|| = \sqrt{x_n^2 + y_n^2} = \frac{1}{\sqrt{2}}$ for all $n \in \mathbb{N}$ and so $((x_n, y_n))$ is a bounded sequence in \mathbb{R}^2 . Hence by the Bolzano-Weierstrass theorem in \mathbb{R}^2 , there exist $(x, y) \in \mathbb{R}^2$ and a convergent subsequence $((x_{n_k}, y_{n_k}))$ of $((x_n, y_n))$ such that $(x_{n_k}, y_{n_k}) \to (x, y)$. Since f is continuous at (x, y), $(n_k, \frac{1}{n_k}) = f(x_{n_k}, y_{n_k}) \to f(x, y) \in \mathbb{R}^2$. Consequently, the sequence (n_k) converges in \mathbb{R} , which is not true, since (n_k) is unbounded. Hence it follows that there cannot exist any sequence $((x_n, y_n))$ in \mathbb{R}^2 such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.

20. Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4 + y^2}$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution: Let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$ and $y_n \to 0$. Since

$$\left| \frac{x_n^3 y_n}{x_n^4 + y_n^2} \right| \le \frac{|x_n|^3 |y_n|}{x_n^4} \le |x_n| + |y_n| \to 0,$$

it follows that $\frac{x_n^3 y_n}{x_n^4 + y_n^2} \to 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0.$$

21. Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution: Let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$, $y_n \to 0$, and hence

$$\frac{x_n^3 - y_n^3}{x_n^2 + y_n^2} = \frac{x_n^3}{x_n^2 + y_n^2} - \frac{y_n^3}{x_n^2 + y_n^2} \le |x_n| + |y_n| \to 0.$$

Consequently $\frac{x_n^3 - y_n^3}{x_n^2 + y_n^2} \to 0$, and therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2} = 0.$$

22. Examine whether

$$\lim_{(x,y)\to(0,0)} |x|e^{-|x|/y^2}$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution: Let $f(x,y) = |x|e^{-|x|/y^2}$ for all $(x,y) \in \mathbb{R}^2$ with $y \neq 0$. We have $(0,\frac{1}{n}) \to (0,0)$ and $(\frac{1}{n^2},\frac{1}{n}) \to (0,0)$. Also, $f(0,\frac{1}{n}) \to 0$ and $f(\frac{1}{n^2},\frac{1}{n}) \to \frac{1}{e}$. Since $\lim_{n \to \infty} f(0,\frac{1}{n}) \neq \lim_{n \to \infty} f(\frac{1}{n^2},\frac{1}{n})$, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in \mathbb{R}).

23. Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^2 - 1}{x^2 + y^3}$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution: Let $f(x,y) = \frac{x^3 + y^2 - 1}{x^2 + y^3}$ for all $(x,y) \in \mathbb{R}^2$ with $x^2 + y^3 \neq 0$. We have $(\frac{1}{n},0) \to (0,0)$ and $(\frac{1}{n},\frac{1}{n}) \to (0,0)$. Also, $f(\frac{1}{n},0) = \frac{1/n^3 - 1}{1/n^2} = \frac{1}{n} - n^2 \to -\infty$ and $f(\frac{1}{n},\frac{1}{n}) = \frac{1/n^3 + 1/n^2 - 1}{1/n^2 + 1/n^3} = 1 + \frac{1/n - 1}{1 + 1/n} \to 1$. Since $f(\frac{1}{n},0) \neq f(\frac{1}{n},\frac{1}{n})$, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in \mathbb{R}).

24. Examine whether

$$\lim_{(x,y)\to(0,0)}\frac{x^2y^2+x^2-1}{x^2+y^3}$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution: Let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$ and $y_n \to 0$. Since $0 \le \frac{\sqrt{x_n^2 y_n^4 + 1} - 1}{x_n^2 + y_n^3} \le \frac{x_n^2 y_n^4}{x_n^2 + y_n^3} \le x_n^2$ if $y_n^2 \to 0$, it follows that

$$\frac{\sqrt{x_n^2 y_n^4 + 1} - 1}{x_n^2 + y_n^3} \to 0.$$

Therefore

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^4+1}-1}{x^2+y^3} = 0.$$

25. Examine whether

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2 + x^2 - 1}{x^2 + y^3}$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution: Let $f(x,y) = \frac{x^2y^2 + x^2 - 1}{x^2 + y^3}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. We have $(\frac{1}{\sqrt{n}}, \frac{1}{n}) \to (0,0)$ and $(\frac{1}{n}, 0) \to (0,0)$. Also, $f(\frac{1}{n}, 0) \to -1$ and $f(\frac{1}{\sqrt{n}}, \frac{1}{n}) \to \frac{1}{2}$. Since

$$\lim_{n \to \infty} f\left(\frac{1}{n}, 0\right) \neq \lim_{n \to \infty} f\left(\frac{1}{\sqrt{n}}, \frac{1}{n}\right),$$

 $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in \mathbb{R}).

26. Examine whether

$$\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$$

exists (in \mathbb{R}) and find its value if it exists (in \mathbb{R}).

Solution: Let $f(x,y,z) = \frac{(x+y+z)^2}{x^2+y^2+z^2}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. We have $(\frac{1}{n},0,0) \to (0,0,0)$ and $(\frac{1}{n},\frac{1}{n},0) \to (0,0,0)$. Also, $f(\frac{1}{n},0,0) = 1$ and $f(\frac{1}{n},\frac{1}{n},0) = 2 \to 2$. Since $\lim_{n\to\infty} f(\frac{1}{n},0,0) \neq \lim_{n\to\infty} f(\frac{1}{n},\frac{1}{n},0)$,

$$\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$$

does not exist (in \mathbb{R}).

27. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x+y & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

Examine whether

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exists (in \mathbb{R}).

Solution: We have $(\frac{1}{n},0) \to (0,0)$ and $(\frac{1}{n},\frac{1}{n}) \to (0,0)$. Also, $f(\frac{1}{n},0) = \frac{1}{n} \to 0$ and $f(\frac{1}{n},\frac{1}{n})=1\to 1$. Since

$$\lim_{n \to \infty} f\left(\frac{1}{n}, 0\right) \neq \lim_{n \to \infty} f\left(\frac{1}{n}, \frac{1}{n}\right),$$

 $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in \mathbb{R}).

28. Let $S \subseteq \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$ and r > 0 be such that $(B_r(x_0) \times B_r(y_0)) \setminus \{(x_0, y_0)\} \subseteq S$. Let $\lim_{x \to x_0} f(x, y)$ exist (in \mathbb{R}) for each $y \in B_r(y_0) \setminus \{y_0\}$, $\lim_{y \to y_0} f(x, y)$ exist (in \mathbb{R}) for each $x \in B_r(x_0) \setminus \{x_0\}$ and $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \in \mathbb{R}$. Show that

$$\lim_{x \to x_0} \left(\lim_{y \to y_0} f(x, y) \right) = \lim_{y \to y_0} \left(\lim_{x \to x_0} f(x, y) \right) = \ell.$$

[$\lim_{x\to x_0} (\lim_{y\to y_0} f(x,y))$ and $\lim_{y\to y_0} (\lim_{x\to x_0} f(x,y))$ are called the iterated limits of f at (x_0,y_0) .]

Solution: Let $\varepsilon > 0$. Since $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$, there exists $\delta \in (0,r)$ such that

$$|f(x,y) - \ell| < \frac{\varepsilon}{2}$$

for all $(x,y) \in B_{\delta}((x_0,y_0)) \setminus \{(x_0,y_0)\}$. Let $g(x) = \lim_{y \to y_0} f(x,y)$ for all $x \in B_r(x_0) \setminus \{x_0\}$ and let $x \in B_{\delta/2}(x_0) \setminus \{x_0\}$. Then there exists $s \in (0, \frac{\delta}{2})$ such that $|f(x,y) - g(x)| < \frac{\varepsilon}{2}$ for all $y \in B_s(y_0) \setminus \{y_0\}$. We choose any $y \in B_s(y_0) \setminus \{y_0\}$. Then

$$0 < \|(x,y) - (x_0,y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \sqrt{\frac{\delta^2}{4} + s^2} < \delta,$$

i.e. $(x, y) \in B_{\delta}((x_0, y_0)) \setminus \{(x_0, y_0)\}$ and hence

$$|g(x) - \ell| \le |g(x) - f(x, y)| + |f(x, y) - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\lim_{x\to x_0} g(x) = \ell$, i.e.

$$\lim_{x \to x_0} \left(\lim_{y \to y_0} f(x, y) \right) = \ell.$$

Similarly, we can show that

$$\lim_{y \to y_0} \left(\lim_{x \to x_0} f(x, y) \right) = \ell.$$

29. Show that

$$\lim_{x \to 0} \left(\lim_{y \to 0} \frac{x^2}{x^2 + y^2} \right) \neq \lim_{y \to 0} \left(\lim_{x \to 0} \frac{x^2}{x^2 + y^2} \right)$$

and hence conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$$

does not exist (in \mathbb{R}).

Solution: For each $x \in \mathbb{R} \setminus \{0\}$,

$$\lim_{y \to 0} \frac{x^2}{x^2 + y^2} = \frac{x^2}{x^2} = 1$$

and for each $y \in \mathbb{R} \setminus \{0\}$,

$$\lim_{x \to 0} \frac{x^2}{x^2 + y^2} = 0.$$

Hence

$$\lim_{x \to 0} \left(\lim_{y \to 0} \frac{x^2}{x^2 + y^2} \right) = \lim_{x \to 0} 1 = 1 \neq 0 = \lim_{y \to 0} \left(\lim_{x \to 0} \frac{x^2}{x^2 + y^2} \right).$$

Using Ex. 15 of Practice Problem Set 2, we can conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$$

does not exist (in \mathbb{R}).

30. Show that

$$\lim_{x \to 0} \left(\lim_{y \to 0} \frac{-x^2 y^2}{x^4 + y^4 + (x - y)^2} \right) = 0 = \lim_{y \to 0} \left(\lim_{x \to 0} \frac{-x^2 y^2}{x^4 + y^4 + (x - y)^2} \right)$$

but that

$$\lim_{(x,y)\to(0,0)} \frac{-x^2y^2}{x^4+y^4+(x-y)^2}$$

does not exist (in \mathbb{R}).

Solution: Let $f(x,y) = \frac{-x^2y^2}{x^4 + y^4 + (x-y)^2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then $\lim_{y \to 0} f(x,y) = 0$

for each $x \in \mathbb{R} \setminus \{0\}$ and $\lim_{x\to 0} f(x,y) = 0$ for each $y \in \mathbb{R} \setminus \{0\}$. Consequently,

$$\lim_{x \to 0} \left(\lim_{y \to 0} f(x, y) \right) = 0 = \lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right).$$

Again, we have $(\frac{1}{n},0) \to (0,0)$ and $(\frac{1}{n},\frac{1}{n}) \to (0,0)$. Also, $f(\frac{1}{n},0) \to 0$ and $f(\frac{1}{n},\frac{1}{n}) \to -1$. Since $\lim_{n\to\infty} f(\frac{1}{n},0) \neq \lim_{n\to\infty} f(\frac{1}{n},\frac{1}{n})$, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (in \mathbb{R}).

31. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x \sin \frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$.

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ and $(\lim_{y\to 0} f(x,y)) = 0$ but that $\lim_{x\to 0} f(x,y)$ does not exist (in \mathbb{R}) if $x\in\mathbb{R}\setminus\{0\}$ and so $\lim_{x\to 0} (\lim_{y\to 0} f(x,y))$ is not defined.

Solution: If $((x_n, y_n))$ is any sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(x_n, y_n) \to (0, 0)$, then $x_n \to 0$ and hence $|f(x_n, y_n)| \le |x_n| \to 0$. Therefore $f(x_n, y_n) \to 0$ and so $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Again, for each $y \in \mathbb{R} \setminus \{0\}$, $\lim_{x\to 0} f(x,y) = \lim_{x\to 0} x \sin \frac{1}{y} = 0$ and so $\lim_{x\to 0} (\lim_{y\to 0} f(x,y)) = \lim_{x\to 0} 0 = 0$.

If $x \in \mathbb{R} \setminus \{0\}$, then $\lim_{y\to 0} f(x,y) = \lim_{y\to 0} x \sin \frac{1}{y}$ which does not exist (in \mathbb{R}) and so $\lim_{x\to 0} (\lim_{y\to 0} f(x,y))$ is not defined.

32. Show that

$$\lim_{(x,y)\to(0,0)} \frac{1}{3x^2 + y^4} = \infty.$$

Solution: Let $((x_n, y_n))$ be any sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$, $y_n \to 0$ and hence $3x_n^2 + y_n^4 \to 0$. If r > 0, then there exists $n_0 \in \mathbb{N}$ such that $3x_n^2 + y_n^4 < \frac{1}{r}$ for all $n \ge n_0$ and so $\frac{1}{3x_n^2 + y_n^4} > r$ for all $n \ge n_0$. Therefore $\frac{1}{3x_n^2 + y_n^4} \to \infty$ and consequently

$$\lim_{(x,y)\to(0,0)} \frac{1}{3x^2 + y^4} = \infty.$$

33. Let I be an open interval in $\mathbb R$ and let $F:I\to\mathbb R^m$ be a differentiable function such that $F(t)\cdot F'(t)=0$ for all $t\in I$. Show that $\|F(t)\|$ is constant for all $t\in I$.

Solution: Since F is differentiable, the function $t \mapsto ||F(t)||^2 = F'(t) \cdot F(t)$ from I to \mathbb{R} is also differentiable, and

$$\frac{d}{dt} \|F(t)\|^2 = F'(t) \cdot F(t) + F(t) \cdot F'(t) = 2F(t) \cdot F'(t) = 0 \quad \text{for all } t \in I.$$

Hence, there exists $c \in \mathbb{R}$ such that

$$||F(t)||^2 = c$$
 for all $t \in I$.

Clearly, $c \geq 0$, and so

$$||F(t)|| = \sqrt{c}$$
 for all $t \in I$.