MA15010H: Multi-variable Calculus

(Practice problem set 3: Hint/Model solution)

July - November, 2025

1. *If $f(x,y) = e^x(x\cos y - y\sin y)$ for all $(x,y) \in \mathbb{R}^2$, then show that $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.*

Solution. For all $(x,y) \in \mathbb{R}^2$, we have $f_x(x,y) = e^x(x\cos y - y\sin y) + e^x \cos y$ and $f_y(x,y) = e^x(-x\sin y - y\cos y - \sin y)$. Hence $f_{xx}(x,y) = e^x(x\cos y - y\sin y) + 2e^x\cos y$ and $f_{yy}(x,y) = e^x(-x\cos y - 2\cos y + y\sin y)$ for all $(x,y) \in \mathbb{R}^2$. Therefore $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$. $\square$

2. *If $f(x,y) = x^2\tan^{-1}\left(\frac{y}{x}\right)$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R} : x = 0\}$, then find $\frac{\partial^2 f}{\partial x \partial y}(1,1)$.*

Solution. For all $(x,y) \in S = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$, we have $\frac{\partial f}{\partial y}(x,y) = \frac{x^3}{x^2+y^2}$ and hence $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$. Therefore $\frac{\partial^2 f}{\partial x \partial y}(1,1) = 1$. $\square$

3. *If $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, then show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ at each point of $\mathbb{R}^3 \setminus \{(0,0,0)\}$.*

Solution. We have $\frac{\partial f}{\partial x}(x,y,z) = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$ and $\frac{\partial^2 f}{\partial x^2}(x,y,z) = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. Similarly, we find that $\frac{\partial^2 f}{\partial y^2}(x,y,z) = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}}$ and $\frac{\partial^2 f}{\partial z^2}(x,y,z) = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. Therefore

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) + \frac{\partial^2 f}{\partial y^2}(x,y,z) + \frac{\partial^2 f}{\partial z^2}(x,y,z) = 0$$

for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. $\square$

4. *If $f(x,y) = \sqrt{|x^2-y^2|}$ for all $(x,y) \in \mathbb{R}^2$, then find all $u \in \mathbb{R}^2$ with $\|u\| = 1$ for which the directional derivative $D_u f(0,0)$ exists (in $\mathbb{R}$).*

Solution. If $u = (u_1,u_2) \in \mathbb{R}^2$ with $\|u\| = 1$, then

$$D_u f(0,0) = \lim_{t\to 0}\frac{f((0,0)+tu) - f(0,0)}{t} = \lim_{t\to 0}\frac{f(tu_1,tu_2) - 0}{t} = \lim_{t\to 0}\frac{|t|\sqrt{|u_1^2-u_2^2|}}{t}$$

exists in $\mathbb{R}$ if $u_1^2 = u_2^2$. Since $\|u\| = \sqrt{u_1^2+u_2^2} = 1$, $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = \pm\frac{1}{\sqrt{2}}$ and $u_2 = \pm\frac{1}{\sqrt{2}}$. Therefore, $D_u f(0,0)$ exists (in $\mathbb{R}$) iff

$$u \in \left\{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\right\}.$$

$\square$

5. *If $f(x,y) = ||x|-|y|| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$, then find all $u \in \mathbb{R}^2$ with $\|u\| = 1$ for which the directional derivative $D_u f(0,0)$ exists (in $\mathbb{R}$).*

Solution. If $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$, then

$$D_u f(0,0) = \lim_{t\to 0} \frac{f((0,0)+tu) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(tu_1, tu_2) - 0}{t} = \lim_{t\to 0} \frac{|t|(||u_1| - |u_2|| - |u_1| - |u_2|)}{t}$$

exists in $\mathbb{R}$ if $|u_1| = |u_2|$, i.e., iff $|u_1| - |u_2| = 0$. If $u_1^2 + u_2^2 = 1$ and hence $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = 0$, i.e., $u_1 = 0$ or $u_2 = 0$. Since $u_1^2 + u_2^2 = 1$, $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = \pm 1$ or else $u_2 = \pm 1$. Therefore, $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u \in \{(1,0), (-1,0), (0,1), (0,-1)\}$. $\square$

6. *Find all $u \in \mathbb{R}^2$ with $\|u\| = 1$ for which the directional derivative $D_u f(0,0)$ exists (in $\mathbb{R}$), if for all $(x,y) \in \mathbb{R}^2$,*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Solution. If $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = \sqrt{u_1^2 + u_2^2} = 1$, then

$$D_u f(0,0) = \lim_{t\to 0} \frac{f((0,0)+tu) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = \lim_{t\to 0} \frac{u_1 u_2}{t(u_1^2 + u_2^2)}$$

exists (in $\mathbb{R}$) iff $u_1 u_2 = 0$, i.e. iff $u_1 = 0$ or $u_2 = 0$. Since $u_1^2 + u_2^2 = 1$, $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = \pm 1$ or else $u_2 = \pm 1$. Therefore $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u \in \{(1,0), (-1,0), (0,1), (0,-1)\}$. $\square$

7. *Find all $u \in \mathbb{R}^2$ with $\|u\| = 1$ for which the directional derivative $D_u f(0,0)$ exists (in $\mathbb{R}$), if for all $(x,y) \in \mathbb{R}^2$,*

$$f(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Solution. Let $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$.

If $u_2 = 0$, then

$$D_u f(0,0) = \lim_{t\to 0} \frac{f((0,0)+tu) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$$

Again, if $u_2 \neq 0$, then

$$D_u f(0,0) = \lim_{t\to 0} \frac{f((0,0)+tu) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = \lim_{t\to 0} \frac{u_1}{tu_2}$$

exists (in $\mathbb{R}$) iff $u_1 = 0$.

Thus, combining the two cases, we find that $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u_2 = 0$ or else $u_1 = 0$. Since $u_1^2 + u_2^2 = 1$, $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = \pm 1$ or else $u_2 = \pm 1$. Therefore $D_u f(0,0)$ exists (in $\mathbb{R}$) iff $u \in \{(1,0), (-1,0), (0,1), (0,-1)\}$. $\square$

8. *State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that $f_x(0,0)$ exists (in $\mathbb{R}$), then $f_y(0,0)$ must exist (in $\mathbb{R}$).*

Solution. Let $f(x,y) = |y|$ for all $(x,y) \in \mathbb{R}^2$. If $(x,y) \in \mathbb{R}^2$ and (x_n, y_n) is any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x,y)$, then $y_n \to y$ and hence $f(x_n, y_n) = |y_n| \to |y| = f(x,y)$. Therefore f is continuous at (x,y) and since $(x,y) \in \mathbb{R}^2$ is arbitrary, $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Also,

$$f_x(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0,$$

but

$$f_y(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{|t|}{t},$$

which does not exist (in $\mathbb{R}$). Therefore the given statement is FALSE. $\square$

9. *State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that for each $u \in \mathbb{R}^2$ with $\|u\| = 1$, the directional derivative of f at $(0,0)$ along u is 0, then f must be continuous at $(0,0)$.*

Solution. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\left(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}\right) \to (0,0)$, but $f\left(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}\right) = 1$ for all $n \in \mathbb{N}$, so that

$$f\left(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}\right) \to 1 \neq 0 = f(0,0).$$

Hence f is not continuous at $(0,0)$.

Again, let $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$. We have

$$f'_u(0,0) = \lim_{t\to 0} \frac{f((0,0)+tu) - f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$$

(The inequalities $tu_2 < t^2u_1^2 < 2tu_2$ are equivalent to the inequalities (i) $u_2 < tu_1^2 < 2u_2$ if $t > 0$ and (ii) $u_2 > tu_1^2 > 2u_2$ if $t < 0$. We can make $|tu_1^2|$ arbitrarily small for sufficiently small $|t| > 0$ and hence for such t, at least one inequality in each of (i) and (ii) cannot be satisfied. Thus we get $f(tu_1, tu_2) = 0$ for sufficiently small $|t| > 0$.) Therefore the given statement is FALSE. $\square$

10. *Let the height $H(x,y)$ of a hill from the ground (considered as the xy-plane) at each point $(x,y) \in (-300,300) \times (-200,200)$ be given by $H(x,y) = 1000 - 0.005x^2 - 0.01y^2$. We assume that the positive x-axis points east and the positive y-axis points north. Consider a person situated at the point $(60,40,966)$ on the hill.*

(a) If the person starts walking due south, then will (s)he start to ascend or descend the hill?

(b) If the person starts walking north-west, then will (s)he start to ascend or descend the hill?

(c) If the person starts climbing further, in which direction will (s)he find it most difficult to climb?

Solution. Let $S = (-300, 300) \times (-200, 200)$. Since $H_x(x, y) = -0.01x$ and $H_y(x, y) = -0.02y$ for all $(x, y) \in S$, $H_x : S \to \mathbb{R}$ and $H_y : S \to \mathbb{R}$ are continuous. Hence $H : S \to \mathbb{R}$ is differentiable and so

$$D_u H(60, 40) = \nabla H(60, 40) \cdot u = H_x(60, 40)u_1 + H_y(60, 40)u_2 = -0.6u_1 - 0.8u_2$$

for all $u = (u_1, u_2) \in \mathbb{R}^2$ with $\|u\| = 1$.

(a) The direction of south corresponds to $u = (0, -1)$ and since $D_u H(60, 40) = 0.8 > 0$, H increases in this direction and hence the person will ascend the hill if he starts walking due south.

(b) The direction of north-west corresponds to $u = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and since

$$D_u H(60, 40) = -\frac{0.2}{\sqrt{2}} < 0,$$

H decreases in this direction and hence the person will descend the hill if he starts walking north-west.

(c) Since H increases fastest in the direction of $u = \nabla H(60, 40) = (-0.6, -0.8)$, the person will find it most difficult to climb the hill in the direction of $(-0.6, -0.8)$.

$\square$

11. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f(x, y) = \begin{cases} \frac{x^2(x-y)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Examine whether $f_{xy}(0, 0) = f_{yx}(0, 0)$.

Solution. We have

$$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_x(0, h) - f_x(0, 0)}{h}, \qquad f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$$

Now,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0, \qquad f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = 0.$$

Also, if $h \in \mathbb{R} \setminus \{0\}$, then

$$f_y(h, 0) = \lim_{k \to 0} \frac{f(h, k) - f(h, 0)}{h} = \lim_{k \to 0} \frac{h^2(h - k)}{k^2 + h^2} = h$$

and if $k \in \mathbb{R} \setminus \{0\}$,

$$f_x(0, k) = \lim_{h \to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h \to 0} \frac{h^2(h - k)}{h^2 + k^2} = 0.$$

Hence $f_{xy}(0, 0) = \lim_{h \to 0} \frac{0-0}{h} = 0$ and $f_{yx}(0, 0) = 1$. Therefore $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. $\square$

12. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by*

$$f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Determine all the points of $\mathbb{R}^2$ where $f_{xy}:\mathbb{R}^2\to\mathbb{R}$ and $f_{yx}:\mathbb{R}^2\to\mathbb{R}$ are continuous.

Solution. For all $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$, we have

$$f_x(x,y)=\frac{x^4y-y^5+4x^2y^3}{(x^2+y^2)^2}$$

and

$$f_{xy}(x,y)=\frac{x^6-y^6+9x^4y^2-9x^2y^4}{(x^2+y^2)^3}.$$

Similarly, for all $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$, we have

$$f_y(x,y)=\frac{x^5-xy^4-4x^3y^2}{x^2+y^2}$$

and

$$f_{yx}(x,y)=\frac{x^6-y^6-9x^4y^2+9x^2y^4}{(x^2+y^2)^3}.$$

Also, we have shown in an example in lectures that $f_{xy}(0,0)=-1$ and $f_{yx}(0,0)=1$. Clearly $f_{xy}:\mathbb{R}^2\to\mathbb{R}$ and $f_{yx}:\mathbb{R}^2\to\mathbb{R}$ are continuous at each point of $\mathbb{R}^2\setminus\{(0,0)\}$. Again, since $\left(\frac{1}{n},0\right)\to(0,0)$ and $(0,\frac{1}{n})\to(0,0)$, but

$$\lim_{n\to\infty}f_{xy}\left(\frac{1}{n},0\right)=1\neq f_{xy}(0,0)$$

and

$$\lim_{n\to\infty}f_{yx}\left(0,\frac{1}{n}\right)=1\neq f_{yx}(0,0),\ f_{xy} \text{ and } f_{yx} \text{ are not continuous at } (0,0).$$

□

13. *Let $f(x,y)=x+y^2+xy$ for all $(x,y)\in\mathbb{R}^2$. Using directly the definition of differentiability, show that $f:\mathbb{R}^2\to\mathbb{R}$ is differentiable and also find $f'(x_0,y_0)$, where $(x_0,y_0)\in\mathbb{R}^2$.*

Solution. Let $(x_0,y_0)\in\mathbb{R}^2$. For all $(h,k)\in\mathbb{R}^2$, we have

$$\begin{aligned}f((x_0,y_0)+(h,k))-f(x_0,y_0)&=f(x_0+h,y_0+k)-f(x_0,y_0)\\&=x_0+h+(y_0+k)^2+(x_0+h)(y_0+k)-x_0-y_0^2-x_0y_0\\&=h+(y_0+k)^2-y_0^2+(x_0+h)(y_0+k)-x_0y_0\\&=h+(y_0^2+2y_0k+k^2-y_0^2)+(x_0y_0+hy_0+x_0k+hk-x_0y_0)\\&=h+2y_0k+k^2+hy_0+x_0k+hk\end{aligned}$$

Let $\alpha=(1+y_0,x_0+2y_0)$. Then $\alpha\in\mathbb{R}^2$ and

$$\lim_{(h,k)\to(0,0)}\frac{f((x_0,y_0)+(h,k))-f(x_0,y_0)-\alpha\cdot(h,k)}{\|(h,k)\|}=\lim_{(h,k)\to(0,0)}\frac{k^2+hk}{\sqrt{h^2+k^2}}=0,$$

since for all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have

$$\frac{|k^2+hk|}{\sqrt{h^2+k^2}} \le \frac{|k|^2+|h||k|}{|k|+|h|}|k| \le 2|k|$$

and since $2|k| \to 0$ as $(h,k) \to (0,0)$. Therefore f is differentiable at (x_0,y_0) and $f'(x_0,y_0) = [1+y_0, x_0+2y_0]$. Since $(x_0,y_0) \in \mathbb{R}^2$ is arbitrary, f is differentiable. □

14. *Let S be a nonempty open subset of $\mathbb{R}^m$ and let $g : S \to \mathbb{R}^m$ be continuous at $x_0 \in S$. If $f : S \to \mathbb{R}$ is such that $f(x) - f(x_0) = g(x)\cdot(x-x_0)$ for all $x \in S$, then show that f is differentiable at x_0.*

Solution. For all $h \in \mathbb{R}^m$ with $x_0+h \in S$, we have

$$f(x_0+h) - f(x_0) = g(x_0+h)\cdot h.$$

Now $g(x_0) \in \mathbb{R}^m$ and for all $h \in \mathbb{R}^m \setminus \{0\}$ with $x_0+h \in S$, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\frac{|f(x_0+h)-f(x_0)-g(x_0)\cdot h|}{\|h\|} &= \frac{|(g(x_0+h)-g(x_0))\cdot h|}{\|h\|}\\ &\le \frac{\|g(x_0+h)-g(x_0)\|\|h\|}{\|h\|}\\ &= \|g(x_0+h)-g(x_0)\|.\end{aligned}$$

Since g is continuous at x_0, $\lim_{\|h\|\to 0}\|g(x_0+h)-g(x_0)\| = 0$ and hence we get

$$\lim_{h\to 0}\frac{|f(x_0+h)-f(x_0)-g(x_0)\cdot h|}{\|h\|} = 0.$$

Therefore f is differentiable at x_0. □

15. *The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions $\left(\frac{1}{\sqrt5},\frac{2}{\sqrt5}\right)$ and $\left(\frac{2}{\sqrt5},\frac{1}{\sqrt5}\right)$ are 1 and 2 respectively. Find $f_x(0,0)$ and $f_y(0,0)$.*

Solution. Since f is differentiable at $(0,0)$, $D_uf(0,0) = \nabla f(0,0)\cdot u = f_x(0,0)u_1 + f_y(0,0)u_2$ for all $u = (u_1,u_2) \in \mathbb{R}^2$ with $\|u\| = 1$. Hence taking $u = \left(\frac{1}{\sqrt5},\frac{2}{\sqrt5}\right)$ and $u = \left(\frac{2}{\sqrt5},\frac{1}{\sqrt5}\right)$ respectively, we get

$$\frac{1}{\sqrt5}f_x(0,0) + \frac{2}{\sqrt5}f_y(0,0) = 1, \qquad \frac{2}{\sqrt5}f_x(0,0) + \frac{1}{\sqrt5}f_y(0,0) = 2.$$

Solving these two equations, we get $f_x(0,0) = \sqrt5$ and $f_y(0,0) = 0$. □

16. *If $f : \mathbb{R}^m \to \mathbb{R}$ satisfies $|f(x)| \le \|x\|^2$ for all $x \in \mathbb{R}^m$, then examine whether f is differentiable at 0.*

Solution. Since $|f(0)| \le \|0\|^2 = 0$, we have $f(0) = 0$. If $\alpha = 0$, then $h \in \mathbb{R}^m$ and for all $h \in \mathbb{R}^m \setminus \{0\}$, we have

$$\frac{|f(h)-f(0)-\alpha h|}{\|h\|} \le \frac{\|h\|^2}{\|h\|} = \|h\|.$$

Hence it follows that
$$\lim_{h\to 0} \frac{|f(h)-f(0)-\alpha h|}{\|h\|} = 0.$$
Therefore f is differentiable at 0. $\square$

17. *Let $f(x) = \|x\|$ for all $x \in \mathbb{R}^n$. Examine whether $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at 0.*

Solution. Since
$$\lim_{t\to 0} \frac{f(0+te_1)-f(0)}{t} = \lim_{t\to 0} \frac{|t|}{t}$$
does not exist (in $\mathbb{R}$), $\frac{\partial f}{\partial x_1}(0)$ does not exist (in $\mathbb{R}$). Consequently f is not differentiable at 0. $\square$

18. *If $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$, then examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.*

Solution. We have $f_x(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0$

and $f_y(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$

Now
$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} \neq 0,$$
since $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$ but
$$\lim_{n\to\infty} \frac{\sqrt{\frac{1}{n^2}}}{\sqrt{\frac{2}{n^2}}} = \frac{1}{\sqrt{2}} \neq 0.$$
Therefore f is not differentiable at $(0,0)$. $\square$

19. *If $f(x,y) = ||x|-|y|| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$, then examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.*

Solution. We have
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$
and
$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0.$$
Now
$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{\sqrt{h^2+k^2}} \neq 0,$$
since $\left(\frac{2}{n}, \frac{1}{n}\right) \to (0,0)$ but
$$\lim_{n\to\infty} \frac{\frac{2}{n}-\frac{1}{n}}{\sqrt{\frac{4}{n^2}+\frac{1}{n^2}}} = \frac{1}{\sqrt{5}} \neq 0.$$
Hence f is not differentiable at $(0,0)$. $\square$

20. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by* $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$
Examine whether f *is differentiable at* $(0,0)$.

Solution. We have $f_x(0,0) = \lim_{t\to 0} \dfrac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \dfrac{t}{t} = 1$ and
$f_y(0,0) = \lim_{t\to 0} \dfrac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \dfrac{0}{t} = 0$. Now,

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\frac{h^3}{h^2+k^2}-h}{\sqrt{h^2+k^2}} \neq 0,$$

since $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$ but

$$\lim_{n\to\infty} \frac{\frac{1}{2n}}{\sqrt{\frac{2}{n^2}}} = \frac{1}{2\sqrt{2}} \neq 0.$$

Therefore f is not differentiable at $(0,0)$. □

21. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & \textit{if } y \neq 0, \\ 0 & \textit{if } y = 0. \end{cases}$$

Examine whether f *is differentiable at* $(0,0)$.

Solution. We have $f_x(0,0) = \lim_{t\to 0} \dfrac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \dfrac{0}{t} = 0$ and
$f_y(0,0) = \lim_{t\to 0} \dfrac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \dfrac{t}{|t|}\dfrac{|t|}{t} = 1$. Now

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\frac{k}{|k|}\sqrt{h^2+k^2}-k}{\sqrt{h^2+k^2}} \neq 0,$$

since $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$ but

$$\lim_{n\to\infty} \frac{\frac{\sqrt{2}}{n} - \frac{1}{n}}{\frac{\sqrt{2}}{n}} = 1 - \frac{1}{\sqrt{2}} \neq 0.$$

Hence f is not differentiable at $(0,0)$. □

22. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be defined as*

$$f(x,y) = \begin{cases} \sqrt{x^2+y^2} & \textit{if } y > 0 \\ x & \textit{if } y = 0 \\ -\sqrt{x^2+y^2} & \textit{if } y < 0 \end{cases}$$

Examine whether f *is differentiable at* $(0,0)$.

Solution. We have $f_x(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x} = \lim\limits_{x\to 0} \dfrac{x}{x} = 1$. Also, since

$$\lim_{y\to 0^+} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0^+} \frac{\sqrt{y^2}}{y} = 1$$

and

$$\lim_{y\to 0^-} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0^-} \frac{-\sqrt{y^2}}{y} = 1,$$

we get $f_y(0,0) = 1$. Now,

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\sqrt{h^2+k^2} - h - k}{\sqrt{h^2+k^2}} \neq 0,$$

since $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$ but

$$\lim_{n\to\infty} \frac{\frac{\sqrt{2}}{n} - \frac{2}{n}}{\frac{\sqrt{2}}{n}} \neq 0.$$

Hence f is not differentiable at $(0,0)$. □

23. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$$

Examine whether f is differentiable at $(0,0)$.

Solution. We have $\left(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}\right) \to (0,0)$ but $f\left(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}\right) = 1 \neq 0 = f(0,0)$. Hence f is not continuous at $(0,0)$ and consequently f is not differentiable at $(0,0)$. □

24. *For all $(x,y) \in \mathbb{R}^2$, let*

$$f(x,y) = \begin{cases} x & \text{if } |x| < |y|, \\ -x & \text{if } |x| \geq |y|. \end{cases}$$

Examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

Solution. We have

$$f_x(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{-t-0}{t} = -1$$

and

$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

Now,

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{f(h,k) + h}{\sqrt{h^2+k^2}}$$

for $(0,0)$, but

$$\frac{|\left(\frac{1}{n},\frac{1}{n}\right)+1|}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}}=\frac{2/n}{\sqrt{2}/n}\to\frac{2}{\sqrt{2}}\neq 0.$$

Therefore f is not differentiable at $(0,0)$. □

25. *Let* $f:\mathbb{R}^2\to\mathbb{R}$ *be defined by*

$$f(x,y)=\begin{cases}\frac{\sin(x^2y^2)}{x^2+y^2} & \textit{if } (x,y)\neq(0,0),\\ 0 & \textit{if } (x,y)=(0,0).\end{cases}$$

Examine whether f *is differentiable at* $(0,0)$.

Solution. We have

$$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$$

and

$$f_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0.$$

For all $(h,k)\in\mathbb{R}^2\setminus\{(0,0)\}$, we have $\epsilon(h,k)=\frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$. This implies that

$$\left|\frac{\sin(h^2k^2)}{(h^2+k^2)\sqrt{h^2+k^2}}\right|\leq\frac{h^2k^2}{(h^2+k^2)\sqrt{h^2+k^2}}=\sqrt{h^2+k^2}.$$

So $\lim_{(h,k)\to(0,0)}\epsilon(h,k)=0$ and so f is differentiable at $(0,0)$. □

26. *Let* $f:\mathbb{R}^2\to\mathbb{R}$ *be defined by*

$$f(x,y)=\begin{cases}\sin^2 x+x^2\sin\frac{1}{x} & \textit{if } x\neq 0,\\ 0 & \textit{if } x=0.\end{cases}$$

Examine whether f *is differentiable at* $(0,0)$.

Solution. We have $f_x(0,0)=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to 0}\frac{\sin^2 t+t^2\sin\frac{1}{t}}{t}=0$ and $f_y(x,y)=0$ for all $(x,y)\in\mathbb{R}^2$. Since $f_y:\mathbb{R}^2\to\mathbb{R}$ is continuous at $(0,0)$, it follows that g is differentiable at $(0,0)$. □

27. *Let* $f:\mathbb{R}^2\to\mathbb{R}$ *be defined by*

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \textit{if } (x,y)\neq(0,0),\\ 0 & \textit{if } (x,y)=(0,0).\end{cases}$$

Show that f *is differentiable at* $(0,0)$ *although neither* $f_x:\mathbb{R}^2\to\mathbb{R}$ *nor* $f_y:\mathbb{R}^2\to\mathbb{R}$ *is continuous at* $(0,0)$.

Solution. Here $f_x(0,0) = f_y(0,0) = 0$. For all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$,
$$\epsilon(h,k) = \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} \leq \sqrt{h^2+k^2},$$
so that
$$\lim_{(h,k)\to(0,0)} \epsilon(h,k) = 0.$$
Hence f is differentiable at $(0,0)$.

Again,
$$f_x(x,y) = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$
for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Now $\left(\frac{2\pi n}{n}, 0\right)$ is a sequence in $\mathbb{R}^2$ converging to $(0,0)$ but
$$f_x\left(\frac{1}{2\pi n}, 0\right) = -1 \text{ for all } n \in \mathbb{N} \text{ and so } f_x\left(\frac{1}{2\pi n}, 0\right) \to -1 \neq f_x(0,0).$$
This shows that f_x is not continuous at $(0,0)$. Similarly f_y is not continuous at $(0,0)$. $\square$

28. *Let*
$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right) & \textit{if } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$
Examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.

Solution. For all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $f_x(x,y) = 2x\cos\left(\frac{1}{x^2+y^2}\right) + \frac{2x}{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right)$. Now $\left(\frac{\sqrt{2}}{(\sqrt{4n+1})\pi}, 0\right) \to (0,0)$ but $f_x\left(\frac{\sqrt{2}}{\sqrt{4n+1}\pi}, 0\right) = \sqrt{2(4n+1)\pi} \to \infty$. Hence $\lim_{(x,y)\to(0,0)} f_x(x,y)$ does not exist (in $\mathbb{R}$) and consequently f_x is not continuous at $(0,0)$. Therefore f is not continuously differentiable. $\square$

29. *Let $\alpha \in \mathbb{R}$ and $\alpha > 0$. If $f(x,y) = |xy|^\alpha$ for all $(x,y) \in \mathbb{R}^2$, then determine all values of α for which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.*

Solution. We have $f_x(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$ and
$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$
For all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, let
$$\varphi(h,k) = \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = \frac{|hk|^\alpha}{\sqrt{h^2+k^2}}.$$
If $\alpha > \frac{1}{2}$, then
$$\varphi(h,k) \leq \frac{(h^2+k^2)^\alpha}{\sqrt{h^2+k^2}} = (h^2+k^2)^{\alpha - \frac{1}{2}},$$
and so $\lim_{(h,k)\to(0,0)} \varphi(h,k) = 0$. Consequently f is differentiable at $(0,0)$.
Again, if $\alpha \leq \frac{1}{2}$, then $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$ but $\varphi\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{\sqrt{2}}n^{1-2\alpha} \neq 0$ (for $\alpha = \frac{1}{2}$, $\varphi\left(\frac{1}{n}, \frac{1}{n}\right) \to$

$\frac{1}{\sqrt{2}}$ and for $\alpha < \frac{1}{2}$, the sequence $\varphi\left(\frac{1}{n}, \frac{1}{n}\right)$ is unbounded). Hence $\lim_{(h,k)\to(0,0)} \varphi(h,k) \neq 0$ and so f is not differentiable at $(0,0)$. $\square$

30. *Let $f(x,y) = |xy|$ for all $(x,y) \in \mathbb{R}^2$. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.*

Solution. Let $S_1 = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 : xy < 0\}$. Then $f(x,y) = xy$ for all $(x,y) \in S_1$ and $f(x,y) = -xy$ for all $(x,y) \in S_2$. Since $f_x(x,y) = y$ and $f_y(x,y) = x$ for all $(x,y) \in S_1$, we find that both $f_x : S_1 \to \mathbb{R}$ and $f_y : S_1 \to \mathbb{R}$ are continuous. Hence f is differentiable at every point of S_1. By a similar argument, we can show that f is differentiable at every point of S_2. If $\alpha(\neq 0) \in \mathbb{R}$, then $f_y(\alpha, 0) = \lim_{t\to 0} \frac{f(\alpha,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{|\alpha||t|}{t}$ does not exist (in $\mathbb{R}$) and similarly $f_x(0,\alpha)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at any point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for which $xy = 0$. Again, $f_x(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$ and

$$\lim_{(h,k)\to(0,0)} \frac{[f(h,k) - f(0,0)] - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} = 0$$

(since $|h||k| \leq h^2 + k^2$ for all $(h,k) \in \mathbb{R}^2$). Hence f is differentiable at $(0,0)$. Therefore, the set of all points of $\mathbb{R}^2$ at which f is differentiable is $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\} \cup \{(0,0)\}$. $\square$

31. *Let $f(x,y) = (xy)^{\frac{2}{3}}$ for all $(x,y) \in \mathbb{R}^2$. Determine all the points of $\mathbb{R}^2$ at which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.*

Solution. Let $S = \{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$. Since $f_x(x,y) = \frac{2}{3}x^{-\frac{1}{3}}y^{\frac{2}{3}}$ and $f_y(x,y) = \frac{2}{3}x^{\frac{2}{3}}y^{-\frac{1}{3}}$ for all $(x,y) \in S$, we find that both $f_x : S \to \mathbb{R}$ and $f_y : S \to \mathbb{R}$ are continuous. Hence f is differentiable at every point of S. If $\alpha(\neq 0) \in \mathbb{R}$, then $f_y(\alpha,0) = \lim_{t\to 0} \frac{f(\alpha,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{\alpha^{\frac{2}{3}}t^{\frac{2}{3}}}{t} = \lim_{t\to 0} \frac{\alpha^{\frac{2}{3}}}{t^{\frac{1}{3}}}$ does not exist (in $\mathbb{R}$) and similarly $f_x(0,\alpha)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at any point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for which $xy = 0$. Again, $f_x(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$, and

$$f_y(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = 0,$$

$$\lim_{(h,k)\to(0,0)} \frac{[f(h,k) - f(0,0)] - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|h|^{\frac{2}{3}}|k|^{\frac{2}{3}}}{\sqrt{h^2+k^2}} = 0$$

(since $|h|^{\frac{2}{3}}|k|^{\frac{2}{3}} \leq (h^2+k^2)^{\frac{2}{3}}$ for all $(h,k) \in \mathbb{R}^2$). Hence f is differentiable at $(0,0)$. Therefore the set of all points of $\mathbb{R}^2$ at which f is differentiable is $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\} \cup \{(0,0)\}$. $\square$

32. *Let $f(x,y) = |x|\sin(x^2+y^2)$ for all $(x,y) \in \mathbb{R}^2$. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.*

Solution. Clearly f is differentiable at all $(x,y) \in \mathbb{R}^2$ for which $x \neq 0$. Let $y_0 \in \mathbb{R}$. Then

$$f_x(0,y_0) = \lim_{x\to 0} \frac{f(x,y_0) - f(0,y_0)}{x} = \lim_{x\to 0} \frac{|x|\sin(x^2+y_0^2)}{x}$$

which exists in $\mathbb{R}$ (and equals 0) iff $y_0 = \pm\sqrt{n\pi}$ for some $n \in \mathbb{N} \cup \{0\}$. Also, $f_y(x, y) = 2|x|y\cos(x^2 + y^2)$ for all $(x, y) \in \mathbb{R}^2$. So f_y is continuous at each point of $\mathbb{R}^2$. Therefore f is differentiable at $(0, y_0)$ iff $y_0 = \pm\sqrt{n\pi}$ for some $n \in \mathbb{N} \cup \{0\}$. $\square$

33. *Determine all the points of* $\mathbb{R}^2$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *is differentiable, if for all* $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = \begin{cases} x^2 + y^2 & \textit{if both } x, y \in \mathbb{Q}, \\ 0 & \textit{otherwise.} \end{cases}$$

Solution. Since $|f(x, y)| \le x^2 + y^2 = \|(x, y)\|^2$ for all $(x, y) \in \mathbb{R}^2$, by Ex.12(a) of Practice Problem Set - 3, f is differentiable at $(0, 0)$.

Let $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. If $(x_0, y_0) \in \mathbb{Q} \times \mathbb{Q}$, then $(x_0 + \frac{\sqrt{2}}{n}, y_0) \to (x_0, y_0)$ but $f(x_0 + \frac{\sqrt{2}}{n}, y_0) \to 0 \ne x_0^2 + y_0^2 = f(x_0, y_0)$. Again if $(x_0, y_0) \notin \mathbb{Q} \times \mathbb{Q}$, then we choose rational sequences (x_n) and (y_n) such that $x_n \to x_0$ and $y_n \to y_0$. Then $(x_n, y_n) \to (x_0, y_0)$ but $f(x_n, y_n) = x_n^2 + y_n^2 \to x_0^2 + y_0^2 \ne 0 = f(x_0, y_0)$. Hence f is not continuous at (x_0, y_0) and consequently f is not differentiable at (x_0, y_0). $\square$

34. *State TRUE or FALSE with justification: If* $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ *and if* $f(x, y) = |xy|$ *for all* $(x, y) \in S$*, then* $f : S \to \mathbb{R}$ *is differentiable.*

Solution. Clearly $(\frac{1}{2}, 0) \in S$. Since $\lim_{t\to 0} \frac{f(\frac{1}{2},t) - f(\frac{1}{2},0)}{t} = \lim_{t\to 0} \frac{\frac{|t|}{2}}{t}$ does not exist (in $\mathbb{R}$), $f_y(\frac{1}{2}, 0)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at $(\frac{1}{2}, 0)$ and so f is not differentiable. Therefore the given statement is FALSE. $\square$

35. *State TRUE or FALSE with justification: There exists a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *which is differentiable only at* $(1, 0)$.

Solution. For all $(x, y) \in \mathbb{R}^2$, let $f(x, y) = \begin{cases} (x-1)^2 + y^2 & \text{if } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

Taking $\alpha = (1, 0) \in \mathbb{R}^2$, we find that

$$\begin{aligned} \lim_{(h,k)\to(0,0)} \frac{[f(1+h, k) - f(1, 0) - hf_x(1, 0) - kf_y(1, 0)]}{\sqrt{h^2 + k^2}} &\le \lim_{(h,k)\to(0,0)} \frac{h^2 + k^2}{\sqrt{h^2 + k^2}} \\ &= \lim_{(h,k)\to(0,0)} \sqrt{h^2 + k^2} \\ &= 0. \end{aligned}$$

Hence f is differentiable at $(1, 0)$.

Again let $(x, y) \in \mathbb{R}^2 \setminus \{(1, 0)\}$. Then $f(x, y) \ne 0$. We can find a sequence (x_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x$. So $(x_n, y) \to (x, y)$ but $f(x_n, y) = 0$ for all $n \in \mathbb{N}$ and so $f(x_n, y) \to 0 \ne f(x, y)$. Hence f is not continuous at (x, y) and so f is not differentiable at (x, y). Thus $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable only at $(1, 0)$. Therefore the given statement is TRUE. $\square$

36. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be differentiable at* $(0, 0)$ *and let* $\lim_{x\to 0} \frac{f(x,-x) - f(x,x)}{x} = 1$. *Find* $f_y(0, 0)$.

Solution. Since f is differentiable at $(0,0)$, we have $\lim_{t\to 0}\frac{f(t,0)-f(0,0)+f(0,t)-f(0,0)}{\sqrt{2t^2}}=0$ and

$$\lim_{t\to 0}\frac{f(t,-t)-f(t,t)-2f_y(0,0)t}{\sqrt{2t^2}}=0,$$

so $\lim_{t\to 0}\frac{f(t,-t)-f(t,t)}{\sqrt{2}|t|}-2f_y(0,0)=0$. Hence

$$2f_y(0,0)=\lim_{t\to 0}\frac{f(t,-t)-f(t,t)}{t}=1$$

and therefore $f_y(0,0)=\frac{1}{2}$. $\square$

37. *Let $f:\mathbb{R}^m\to\mathbb{R}$ be differentiable at 0 and let $f(\alpha x)=\alpha f(x)$ for all $x\in\mathbb{R}^m$ and for all $\alpha\in\mathbb{R}$. Show that $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}^m$.*

Solution. We have $f(0)=f(0\cdot 0)=0\cdot f(0)=0$. Since f is differentiable at 0, there exists $a\in\mathbb{R}^m$ such that $\lim_{|h|\to 0}\frac{|f(h)-a\cdot h|}{||h||}=\lim_{|h|\to 0}\frac{|f(0+h)-f(0)-a\cdot h|}{||h||}=0$. If $x\in\mathbb{R}^m\setminus\{0\}$, then from above,

$$\lim_{|t|\to 0}\frac{|f(tx)-ta\cdot x|}{||tx||}=0,$$

which gives $\lim_{|t|\to 0}\frac{|f(x)-ta\cdot x|}{|t|||x||}=0$ and so

$$\lim_{|t|\to 0}\frac{|f(x)-a\cdot x|}{|t|||x||}=0$$

and so $|f(x)-a\cdot x|=0$ and hence $f(x)=a\cdot x$.
Since $f(0)=0=a\cdot 0$, we have $f(x)=a\cdot x$ for all $x\in\mathbb{R}^m$. Now, if $x,y\in\mathbb{R}^m$, then $f(x+y)=a\cdot(x+y)=a\cdot x+a\cdot y=f(x)+f(y)$. $\square$

38. *Let $f:\mathbb{R}^m\to\mathbb{R}$ be differentiable at 0 and $f(0)=0$. Show that there exist $\alpha>0$ and $r>0$ such that $|f(x)|\le\alpha||x||$ for all $x\in\mathbb{R}^m$ with $||x||<r$.*

Solution. Since f is differentiable at 0 and $f(0)=0$, there exists $a\in\mathbb{R}^m$ such that

$$\lim_{x\to 0}\frac{|f(x)-a\cdot x|}{||x||}=0.$$

Hence there exists $r>0$ such that $\frac{|f(x)-a\cdot x|}{||x||}<1$ for all $x\in\mathbb{R}^m$ with $0<||x||<r$. Therefore if $x\in\mathbb{R}^m$ with $||x||<r$, then $|f(x)-a\cdot x|\le||x||$ and so $|f(x)|\le|f(x)-a\cdot x|+|a\cdot x|\le||x||+||a||\,||x||=\alpha||x||$, where $\alpha=1+||a||>0$. $\square$

39. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be such that f_x exists (in $\mathbb{R}$) at all points of $B_\delta((x_0,y_0))$ for some $(x_0,y_0)\in\mathbb{R}^2$ and $\delta>0$, f is continuous at (x_0,y_0) and $f_y(x_0,y_0)$ exists (in $\mathbb{R}$). Show that f is differentiable at (x_0,y_0).*

Solution. For all $(h,k)\in B_\delta((0,0))$, we have $f(x_0+h,y_0+k)-f(x_0,y_0)=f(x_0+h,y_0+k)-f(x_0,y_0+k)+f(x_0,y_0+k)-f(x_0,y_0)$.
Now, by the mean value theorem for single real variable, we get $f(x_0+h,y_0+k)-f(x_0,y_0+k)=hf_x(x_0+\theta h,y_0+k)$ for some $\theta\in(0,1)$.

Again, if $\epsilon(k) = f(x_0, y_0 + k) - f(x_0, y_0) - kf_y(x_0, y_0)$ for all $k \in \mathbb{R} \setminus \{0\}$ with $|k| < \delta$ and $\epsilon(0) = 0$, then

$$f(x_0, y_0 + k) - f(x_0, y_0) = kf_y(x_0, y_0) + k\epsilon(k)$$

for all $k \in \mathbb{R}$ with $|k| < \delta$ and $\epsilon(k) \to 0$ as $k \to 0$.
Now,

$$\begin{aligned}
&\lim_{(h,k)\to(0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - hf_x(x_0, y_0) - kf_y(x_0, y_0)}{\sqrt{h^2 + k^2}} \\
&\le \lim_{(h,k)\to(0,0)} \left(\frac{|h|}{\sqrt{h^2 + k^2}} |f_x(x_0 + \theta h, y_0 + k) - f_x(x_0, y_0)| + \frac{|k|}{\sqrt{h^2 + k^2}} |\epsilon(k)| \right) \\
&\le \lim_{(h,k)\to(0,0)} (|f_x(x_0 + \theta h, y_0 + k) - f_x(x_0, y_0)| + |\epsilon(k)|) = 0.
\end{aligned}$$

Therefore f is differentiable at (x_0, y_0). $\square$

40. *Let $f, g : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in S^0$. Show that $f + g : S \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and $\nabla(f + g)(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$.*

Solution. Since f and g are differentiable at $\mathbf{x}_0$, $\nabla f(\mathbf{x}_0), \nabla g(\mathbf{x}_0) \in \mathbb{R}^m$ and by increment theorem, there exist $\delta_1, \delta_2 > 0$ and functions $\varepsilon_1 : B_{\delta_1}(0) \to \mathbb{R}$, $\varepsilon_2 : B_{\delta_2}(0) \to \mathbb{R}$ such that $\lim\limits_{h\to 0} \varepsilon_1(h) = \lim\limits_{h\to 0} \varepsilon_2(h) = 0$ and $f(\mathbf{x}_0 + h) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot h + ||h||\varepsilon_1(h)$ for all $h \in B_{\delta_1}(0)$ and

$$g(\mathbf{x}_0 + h) = g(\mathbf{x}_0) + \nabla g(\mathbf{x}_0) \cdot h + ||h||\varepsilon_2(h) \text{ for all } h \in B_{\delta_2}(0).$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and

$$(f+g)(\mathbf{x}_0+h) = f(\mathbf{x}_0+h)+g(\mathbf{x}_0+h) = (f+g)(\mathbf{x}_0)+(\nabla f(\mathbf{x}_0)+\nabla g(\mathbf{x}_0))\cdot h+||h||[\varepsilon_1(h)+\varepsilon_2(h)]$$

for all $h \in B_\delta(0)$, where $\varepsilon : B_\delta(0) \to \mathbb{R}$ is defined by $\varepsilon(h) = \varepsilon_1(h) + \varepsilon_2(h)$ for all $h \in B_\delta(0)$ and so $\lim_{h\to 0} \varepsilon(h) = \lim_{h\to 0} \varepsilon_1(h) + \lim_{h\to 0} \varepsilon_2(h) = 0$. Therefore by increment theorem, $f + g$ is differentiable at $\mathbf{x}_0$ and $\nabla(f + g)(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$. $\square$

41. *Using the linearization of a suitable function at a suitable point, find an approximate value of $((3.8)^2 + 2(2.1)^2)^{\frac{5}{8}}$.*

Solution. Let $S = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and let $f(x, y) = (x^2 + 2y^2)^{\frac{5}{8}}$ for all $(x, y) \in S$. Then $f_x(x, y) = \frac{5}{4}x(x^2 + 2y^2)^{-\frac{3}{8}}$ and $f_y(x, y) = \frac{5}{2}y(x^2 + 2y^2)^{-\frac{3}{8}}$ for all $(x, y) \in S$. Since $f_x, f_y : S \to \mathbb{R}$ are continuous, $f : S \to \mathbb{R}$ is differentiable. Hence the linearization of f at $(4, 2) \in S$ is given by

$$L(x, y) = f(4, 2) + f_x(4, 2)(x - 4) + f_y(4, 2)(y - 2) = 2 + \frac{1}{10}(x - 4) + \frac{3}{10}(y - 2)$$

for all $(x, y) \in S$. Therefore an approximate value of $f(3.8, 2.1)$ is given by

$$L(3.8, 2.1) = 2 - 0.02 + 0.03 = 2.01.$$

$\square$

42. *Show that the maximum error in calculating the volume of a right circular cylinder is approximately* $\pm 8\%$ *if its radius can be measured with a maximum error of* $\pm 3\%$ *and its height can be measured with a maximum error of* $\pm 2\%$.

Solution. We know that the volume of a right circular cylinder of radius r and height h is given by $V(r,h) = \pi r^2 h$. If $S = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$, then $V : S \to \mathbb{R}$ is differentiable (since $V_r, V_h : S \to \mathbb{R}$ are continuous) and the linearization of V at any point $(r_0, h_0) \in S$ is given by

$$\begin{aligned} L(r,h) &= V(r_0,h_0) + V_r(r_0,h_0)(r - r_0) + V_h(r_0,h_0)(h - h_0) \\ &= V(r_0,h_0) + 2\pi r_0 h_0 (r - r_0) + \pi r_0^2 (h - h_0) \end{aligned}$$

Hence the absolute value of an approximate percentage error in $V(r,h)$ at (r_0,h_0) is given by $\left|\frac{L(r,h)-V(r_0,h_0)}{V(r_0,h_0)}\right| \times 100$. Since it is given that $\left|\frac{r-r_0}{r_0}\right| \times 100 \le 3$ and $\left|\frac{h-h_0}{h_0}\right| \times 100 \le 2$, we get

$$\left|\frac{L(r,h) - V(r_0,h_0)}{V(r_0,h_0)}\right| \times 100 \le 2\left|\frac{r - r_0}{r_0}\right| \times 100 + \left|\frac{h - h_0}{h_0}\right| \times 100 \le 6 + 2 = 8.$$

Therefore the maximum error in calculating $V(r,h)$ at any $(r_0,h_0) \in S$ is approximately $\pm 8\%$. $\square$