MA15010H: Multi-variable Calculus

(Practice problem set 3: Directional derivatives and differentiability)
July - November, 2025

1. If $f(x,y) = e^{x(x\cos y - y\sin y)}$ for all $(x,y) \in \mathbb{R}^2$, then show that $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.
2. If $f(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right)$ for all $(x,y) \in \{(x,y) \in \mathbb{R} : x \neq 0\}$, then find $\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(1,1)}$.
3. If $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, then show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$
at each point of $\mathbb{R}^3 \setminus \{(0,0,0)\}$.
4. Find all $u \in \mathbb{R}^2$ with $\|u\| = 1$ for which the directional derivative $D_u f(0,0)$ exists (in $\mathbb{R}$), if for all $(x,y) \in \mathbb{R}^2$,
   (a) $f(x,y) = \sqrt{|x^2 - y^2|}$.
   (b) $f(x,y) = ||x| - |y|| - |x| - |y|$.
   (c) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   (d) $f(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$
5. State TRUE or FALSE with justification for each of the following statements.
   (a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that $f_x(0,0)$ exists (in $\mathbb{R}$), then $f_y(0,0)$ must exist (in $\mathbb{R}$).
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that for each $u \in \mathbb{R}^2$ with $\|u\| = 1$, the directional derivative of f at $(0,0)$ along u is 0, then f must be continuous at $(0,0)$.
6. Let the height $H(x,y)$ of a hill from the ground (considered as the xy-plane) at each point $(x,y) \in (-300,300) \times (-200,200)$ be given by $H(x,y) = 1000 - 0.005x^2 - 0.01y^2$. We assume that the positive x-axis points east and the positive y-axis points north. Consider a person situated at the point $(60,40,966)$ on the hill.
   (a) If the person starts walking due south, then will (s)he start to ascend or descend the hill?
   (b) If the person starts walking north-west, then will (s)he start to ascend or descend the hill?
   (c) If the person starts climbing further, in which direction will (s)he find it most difficult to climb?
7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} \frac{x^2(x-y)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$
Examine whether $f_{xy}(0,0) = f_{yx}(0,0)$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Determine all the points of $\mathbb{R}^2$ where $f_{xy} : \mathbb{R}^2 \to \mathbb{R}$ and $f_{yx} : \mathbb{R}^2 \to \mathbb{R}$ are continuous.

9. Let $f(x,y) = x + y^2 + xy$ for all $(x,y) \in \mathbb{R}^2$. Using directly the definition of differentiability, show that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and also find $f'(x_0, y_0)$, where $(x_0, y_0) \in \mathbb{R}^2$.

10. Let S be a nonempty open subset of $\mathbb{R}^m$ and let $g : S \to \mathbb{R}^m$ be continuous at $x_0 \in S$. If $f : S \to \mathbb{R}$ is such that $f(x) - f(x_0) = g(x) \cdot (x - x_0)$ for all $x \in S$, then show that f is differentiable at x_0.

11. The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ are 1 and 2 respectively. Find $f_x(0,0)$ and $f_y(0,0)$.

12. Examine the differentiability of f at 0, where
   (a) $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(x)| \le \|x\|^2$ for all $x \in \mathbb{R}^n$.
   (b) $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(x) = \|x\|$ for all $x \in \mathbb{R}^n$.
   (c) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = \sqrt{|xy|}$ for all $x, y \in \mathbb{R}^2$.
   (d) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $f(x,y) = ||x| - |y|| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$.
   (e) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (f) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

   (g) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \sqrt{x^2+y^2} & \text{if } y > 0, \\ x & \text{if } y = 0, \\ -\sqrt{x^2+y^2} & \text{if } y < 0. \end{cases}$$

   (h) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$$

   (i) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} x & \text{if } |x| < |y|, \\ -x & \text{if } |x| \ge |y|. \end{cases}$$

(j) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{\sin(x^2y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(k) $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \sin^2 x + x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2) \sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$ although neither $f_x : \mathbb{R}^2 \to \mathbb{R}$ nor $f_y : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$.

14. Let $f(x,y) = \begin{cases} (x^2+y^2) \cos\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$ Examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.

15. Let $f(x,y) = |xy|^\alpha$ for all $(x,y) \in \mathbb{R}^2$, then determine all values of α for which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

16. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, where for all $(x,y) \in \mathbb{R}^2$,
   (a) $f(x,y) = |xy|$
   (b) $f(x,y) = (xy)^{2/3}$
   (c) $f(x,y) = |x| \sin(x^2+y^2)$
   (d) $f(x,y) = \begin{cases} x^2+y^2 & \text{if both } x,y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

17. State TRUE or FALSE with justification for each of the following statements.
   (a) If $S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ and if $f(x,y) = |xy|$ for all $(x,y) \in S$, then $f : S \to \mathbb{R}$ is differentiable.
   (b) There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is differentiable only at $(1,0)$.

18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(0,0)$ and let $\lim_{x\to 0} \frac{f(x,-x)-f(x,x)}{x} = 1$. Find $f_y(0,0)$.

19. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable at 0 and let $f(\alpha x) = \alpha f(x)$ for all $x \in \mathbb{R}^m$ and for all $\alpha \in \mathbb{R}$. Show that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^m$.

20. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable at 0 and $f(0) = 0$. Show that there exist $\alpha > 0$ and $r > 0$ such that $|f(x)| \leq \alpha \|x\|$ for all $x \in \mathbb{R}^m$ with $\|x\| < r$.

21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that f_x exists (in $\mathbb{R}$) at all points of $B_\delta((x_0,y_0))$ for some $(x_0,y_0) \in \mathbb{R}^2$ and f_x is continuous at (x_0,y_0), $f_y(0,0)$ exists (in $\mathbb{R}$). Show that f is differentiable at (x_0,y_0).

22. Let $f, g : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $x_0 \in S^0$. Show that
   (a) $f+g : S \to \mathbb{R}$ is differentiable at x_0 and $\nabla(f+g)(x_0) = \nabla f(x_0) + \nabla g(x_0)$.
   (b) $fg : S \to \mathbb{R}$ is differentiable at x_0 and $\nabla(fg)(x_0) = g(x_0)\nabla f(x_0) + f(x_0)\nabla g(x_0)$.

(c) If $g(x_0) \neq 0$, then $\frac{f}{g} : S \to \mathbb{R}$ is differentiable at x_0 and
$$\nabla \left(\frac{f}{g} \right) (x_0) = \frac{g(x_0) \nabla f(x_0) - f(x_0) \nabla g(x_0)}{g(x_0)^2}.$$

23. Using the linearization of a suitable function at a suitable point, find an approximate value of $((3.8)^2 + 2(2.1)^3)^{\frac{4}{5}}$.
24. Show that the maximum error in calculating the volume of a right circular cylinder is approximately $\pm 8\%$ if its radius can be measured with a maximum error of $\pm 3\%$ and its height can be measured with a maximum error of $\pm 2\%$.