MA15010H: Multi-variable Calculus

(Practice problem set 4: Higher order derivatives, maxima, minima)

July - November, 2025

1. Let $f(\mathbf{x}) = \|\mathbf{x}\|_2^5$ for all $\mathbf{x} \in \mathbb{R}^m$. Using chain rule, show that $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable and determine $f'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and let $u(x, y, z) = f(x - y, y - z, z - x)$ for all $(x, y, z) \in \mathbb{R}^2$. Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ at each point of $\mathbb{R}^3$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable and let $u(r, \theta) = f(r\cos\theta, r\sin\theta)$ for all $r > 0$, $\theta \in \mathbb{R}$. Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ at each point $(x, y) = (r\cos\theta, r\sin\theta)$ of $\mathbb{R}^2 \setminus \{(0, 0)\}$.

4. Show that a differentiable function $f : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree $\alpha \in \mathbb{R}$ (i.e., $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{0\}$ and for all $t > 0$) iff $\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{0\}$.
   (The only if part of this result is known as Euler's theorem on homogeneous functions.)

5. If $f(x, y) = \tan^{-1}\left(\frac{x^2+y^2}{x-y}\right)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$, where $S = \{(x, y) : x = y\}$, then using Euler's theorem on homogeneous functions, show that $xf_x(x, y) + yf_y(x, y) = \sin(2f(x, y))$ for all $(x, y) \in \mathbb{R}^2 \setminus S$.

6. If $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ is a twice continuously differentiable homogeneous function of degree $n \in \mathbb{R}$, then show that
   $$\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right)(x, y) = n(n-1)f(x, y)$$
   for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $f_x(a, b) = f_y(a, b)$ for all $(a, b) \in \mathbb{R}^2$ and $f(a, 0) > 0$ for all $a \in \mathbb{R}$. Show that $f(a, b) > 0$ for all $(a, b) \in \mathbb{R}^2$.

8. Let $\alpha > 0$ and let $f : \mathbb{R}^m \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq \alpha\|x - y\|^2$ for all $x, y \in \mathbb{R}^m$. Show that f is a constant function.

9. Let S be a nonempty open and convex set in $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be such that $f_x(x, y) = 0 = f_y(x, y)$ for all $(x, y) \in S$. Show that f is a constant function.
   (A set $S \subseteq \mathbb{R}^m$ is called convex if $(1-t)x + ty \in S$ for all $x, y \in S$ and for all $t \in [0, 1]$.)

10. Find the equations of the tangent plane and the normal line to the surface given by $z = x^2 + y^2 - 2xy + 3y - x + 4$ at the point $(2, -3, 18)$.

11. Find all points on the paraboloid $z = x^2 + y^2$ at which the tangent plane to the paraboloid is parallel to the plane $x + y + z = 1$. Also, determine the equations of the corresponding tangent planes.

12. Determine all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^2 \to \mathbb{R}$, if for all $(x, y) \in \mathbb{R}^2$,
    (a) $f(x, y) = x^3 + y^3 - 63x - 63y + 12xy$
    (b) $f(x, y) = 2x^4 + 2x^2y + y^2$
    (c) $f(x, y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$
13. If $f(x, y, z) = x^2 + y^2 + z^2 + 2xyz - 4xz - 2yz - 2x - 4y + 4z$ for all $(x, y, z) \in \mathbb{R}^3$, then find all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^3 \to \mathbb{R}$.
14. If $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, then determine $\max\{x^2 + 2x + y^2 : (x, y) \in S\}$ and $\min\{x^2 + 2x + y^2 : (x, y) \in S\}$.
15. Find the (absolute) maximum value of $f(x, y, z) = 8xyz^2 - 200(x + y + z)$ subject to the constraint $x + y + z = 100, x \geq 0, y \geq 0, z \geq 0$.