MA15010H: Multi-variable Calculus

(Practice problem set 5: Riemann Integration, Fubini's Theorem)

September - November, 2025

(1) Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be defined by $f(x,y) = \varphi(x)\psi(y)$, where $\varphi : [a,b] \to \mathbb{R}$ and $\psi : [c,d] \to \mathbb{R}$ are continuous. Show that

$$\iint_D f(x,y)dxdy = \left(\int_a^b \varphi(x)dx\right)\left(\int_c^d \psi(x)dx\right).$$

(2) Let $f : D = [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}^c \cap [0,1]; \\ 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \text{ and } y \in \mathbb{Q}^c \cap [0,1]; \\ 1 - \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest term and } y \in \mathbb{Q} \cap [0,1]. \end{cases}$$

Then f is integrable and $\iint\limits_D f(x,y)dxdy = 1$. Does repeated integral $\int\limits_0^1 \left(\int\limits_0^1 f(x,y)dy\right) dx$ exist?

(3) Find the volume of the tetrahedron T bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $x - y - z = -1$.

(4) Evaluate the following iterated integrals applying Fubini's Theorem.

(a) $\int\limits_0^1 \int\limits_{x=y}^1 \cos(x^2)dxdy$.

(b) $\int\limits_0^1 \int\limits_{y=\sqrt{x}}^1 e^{y^3}dydx$.

(c) $\int\limits_0^1 \int\limits_{y=x^2}^1 x^3e^{y^3}dydx$.

(d) $\int\limits_0^1 \int\limits_{x=y}^1 \frac{1}{1+x^4}dxdy$.

(e) $\int_0^1 (\tan^{-1}\pi x - \tan^{-1} x)dx$.

(5) Let D be the region lying below the curve $y = \cos x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and above the X-axis. Evaluate $\iint\limits_D \sin x dxdy$.

(6) Let D be the region in $\mathbb{R}^2$ bounded by the curves $y = 2x^2$ and $y = 1+x^2$. Evaluate the double integral $\iint\limits_D (2x^2 + y)dxdy$.

(7) Evaluate $\iint\limits_D x\cos\left(y - \frac{y^3}{3}\right) dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.

(8) Find the volume of the solid enclosed by the surfaces $z = 6 - x^2 - y^2,\ z = 2x^2 + y^2 - 1,\ x = -1,\ x = 1,\ y = -1$ and $y = 1$.

(9) Let D be the solid bounded by the surfaces $y = x^2,\ y = 3x,\ z = 0$ and $z = x^2+y^2$. Find the volume of the solid.

(10) Let D be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z = 1$ and $z = 0$. Find the volume of the solid.