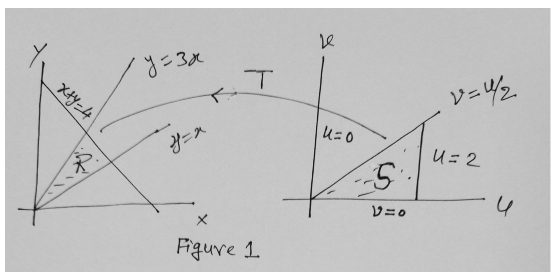
MA15010H: Multi-variable Calculus

(Practice problem set 6 Hint/Model solutions: Change of variables, triple integral)

September - November, 2025

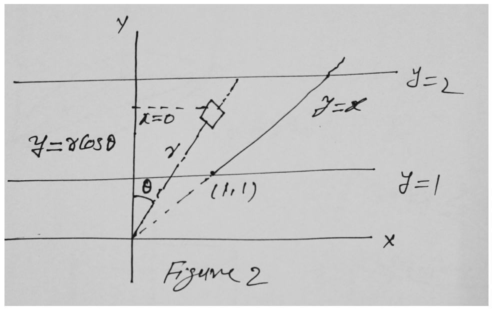
1. Consider the transformation $T : [0, 2\pi]\times[0, 1] \to \mathbb{R}^2$ given by $T(u, v) = (2v\cos u, v\sin u)$.
   (a) For a fixed $v_o \in [0, 1]$, describe the set $\{T(u, v_o) :\ u \in [0, 2\pi]\}$.
   (b) Describe the set $\{T(u, v) :\ [0, 2\pi] \times [0, 1]\}$.
   Solution: (a) If $x = 2v_o \cos u$ and $y = v_o \sin u$, then $\frac{x^2}{4}+\frac{y^2}{1} = v_o^2$. The set $\{T(u, v_o) :\ u \in [0, 2\pi]\}$ is an ellipse.
   (b) The set is the region enclosed by $\frac{x^2}{4} + \frac{y^2}{1} = 1$.
2. Let R be the region in $\mathbb{R}^2$ bounded by the straight lines $y = x,\ y = 3x$ and $x + y = 4$. Consider the transformation $T(u, v) = (u-v, u+v)$. Find the set S satisfying $T(S) = R$.
   Solution: If $x = u - v$ and $y = u + v$, then $y = x$ is mapped to $v = 0$ and $y = 3x$ is mapped to $v = \frac{u}{2}$. The line $x + y = 4$ is mapped to $u = 2$. Please see Figure 1.



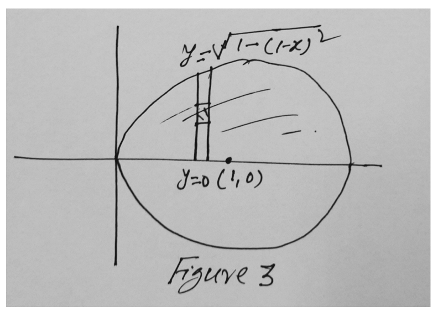
3. Evaluate $\iint\limits_R x\,dxdy$ where R is the region $1 \le x(1 - y) \le 2$ and $1 \le xy \le 2$.
   Solution: Let $u = x(1 - v)$ and $v = xy$. Since $xy \neq 0$, we can solve as $x = u + v$ and $y = \frac{v}{u+v}$. Here $J(u, v) = \frac{1}{u+v}$. The required integral is $\int\limits_1^2\int\limits_1^2 (u + v)\frac{1}{|u+v|}dudv = 1$
4. Evaluate
   (a) $\int\limits_0^{\frac{1}{\sqrt{2}}} \int\limits_{x=y}^{\sqrt{1-y^2}} (x + y)dxdy.$
   (b) $\int\limits_1^2 \int\limits_{x=0}^{y} \frac{1}{(x^2+y^2)^{\frac{3}{2}}}dxdy.$
   (c) $\int\limits_0^2 \int\limits_{y=0}^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}dydx.$
   Solution: (a) The given integral is $\iint\limits_D (x + y)dxdy$, where D is the region bounded by $y = 0,\ y = x$ and the circle $x^2 + y^2 = 1$. By polar coordinates

$$\iint\limits_D (x + y)dxdy = \int_0^{\frac{\pi}{4}} \int_0^1 r(\cos\theta + \sin\theta)rdrd\theta.$$

(b) Please see Figure 2.



By polar coordinate, the given integral becomes $\int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\sec\theta} \frac{1}{r^3} r dr d\theta$.
(c) Please see Figure 3.



The given integral becomes $\iint\limits_D \sqrt{x+y}\,dxdy$, where D is the region in the first quadrant bounded by the circle $(x-1)^2+y^2=1$ and the x-axis. Using polar coordinate, the circle $(x-1)^2+y^2=1$ can be represented by $r=2\cos\theta$. Hence the required integral is $\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 dr d\theta$.

5. Using change of variables $u=x+y$ and $v=x-y$, show that

$$\int_0^1 \int_{y=0}^{y=x} (x-y)dydx = \int_0^1 \int_{u=v}^{2-v} \frac{v}{2} dudv.$$

Solution: We have $u+v=2x$ and $u-v=2y$. The line $x=y$ is mapped to $v=0$ and $x=1$ to $u+v=2$. The x-axis is mapped to $v=u$. Here $J(u,v)=\frac{1}{2}$. Please see the Figure 4.

6. Find the volume of the solid in the first octant bounded below by the surface $z=\sqrt{x^2+y^2}$ above by $x^2+y^2+z^2=8$ as well as the planes $y=0$ and $y=x$.

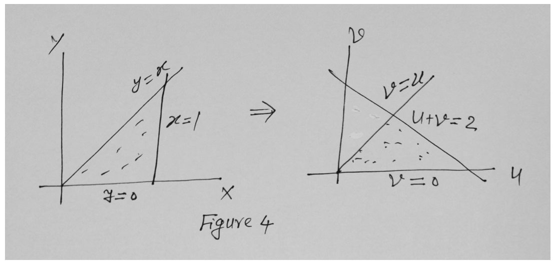


Figure 4

Solution: The given solid lies above the region D, where D is in the first quadrant in $\mathbb{R}^2$ bounded by the circle $x^2+y^2=4$ and the line $y=x$ and $y=0$. Please see Figure 5.

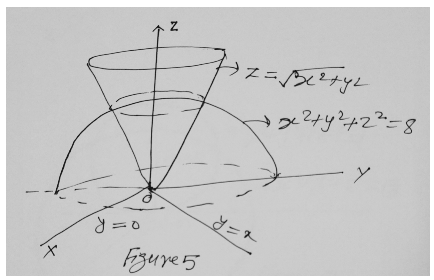
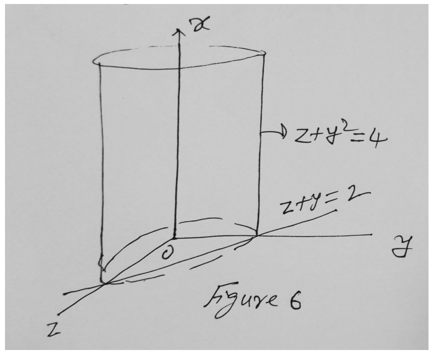


Figure5

Therefore the required volume is given by $\iint\limits_D (\sqrt{8-x^2-y^2}-\sqrt{x^2+y^2})dxdy = \int_0^{\frac{\pi}{4}}\int_0^2(\sqrt{8-r^2}-r)rdrd\theta$.

7. Find the volume of the solid bounded by the surfaces $z=3(x^2+y^2)$ and $z=4-(x^2+y^2)$.
   Solution: The intersection of the surfaces is the set $\{(x,y,3): x^2+Y^2=1\}$. Therefore the volume is given by $\iint\limits_D (4-x^2-y^2-3(x^2+y^2)dxdy$, where D is the region in $\mathbb{R}^2$ enclosed by the circle $x^2+y^2=1$. By polar coordinate the integral becomes $\int_0^{2\pi}\int_0^1(4-4r^2)rdrd\theta$.
8. Let D denote the solid bounded by surfaces $y=x,\ y=x^2,\ z=x$ and $z=0$. Evaluate $\iiint\limits_D ydxdydz$.
   Solution: The projection of the solid D on the xy-plane is give by $R=\{(x,y):\ 0\le x\le 1,\ x^2\le y\le x\}$. The solid D lies above the surface $z=f_1(x,y)=0$ and below $z=f_2(x,y)=x$. Therefore, $\iiint\limits_D ydxdydz = \int_{x=0}^1\left(\int_{y=x^2}^x\left(\int_{z=0}^x ydz\right)dy\right)dx$.
9. Let D denote the solid bounded below by the plane $z+y=2$, above by the cylinder $z+y^2=4$ and on the sides $x=0$ and $x=2$. Evaluate $\iiint\limits_D xdxdydz$.
   Solution: Please see Figure 6.

   Solving $4-y^2=2-y$ implies $y=-1,2$. The projection of the solid D on the xy-plane is given by $R=[0,2]\times[-1,2]$. The solid lies above the surface $z=f_1(x,y)=2-y$



and below $z = f_2(x,y) = 4 - y^2$. Therefore

$$\iiint\limits_D x dx dy dz = \iint\limits_R \left(\int_{z=2-y}^{4-y^2} x dz \right) dx dy = \int_{x=0}^{2} \int_{y=-1}^{2} \int_{z=2-y}^{4-y^2} x dz dy dx.$$

10. Let $D = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \leq 1\}$ and $E = \{(u,v,w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \leq 1\}$. Show that $\iiint\limits_D dx dy dz = \iiint\limits_E 24 du dv dw$.

    Solution: Note that the transformation $T(u,v,w) = (2u, 3v, 4w) = (x,y,z)$ maps E onto D and $J(u,v,w) = 24$.

11. Let D be the solid that lies inside the cylinder $x^2 + y^2 = 1$, below the cone $z = \sqrt{4(x^2+y^2)}$ and above the plane $z = 0$. Evaluate $\iiint\limits_D x^2 dx dy dz$.

    Solution: The projection of the solid D on the xy-plane is $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. By changing to the cylindrical coordinates, the solid D is bounded by $z = 0$ and $z = 2r$. Therefore

$$\iiint\limits_D x^2 dx dy dz = \int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 \cos^2 \theta r dz dr d\theta.$$

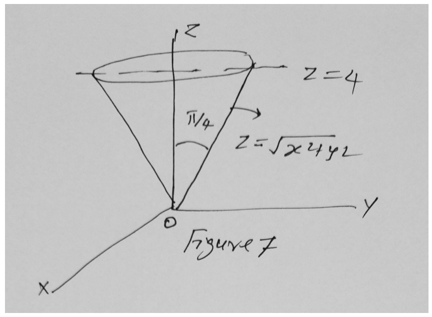
12. Evaluate $\int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int\limits_{x^2+y^2}^{4} x dz dy dx$.

    Solution: Note that $\int\limits_{-2}^{2} \int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int\limits_{x^2+y^2}^{4} x dz dy dx = \iiint\limits_D x dx dy dz$, where where D is the solid bounded below by $z = x^2 + y^2$ and above by $z = 4$. The projection of the solid D on the xy-plane is given by $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$. By the cylindrical coordinates

$$\iint\limits_D x dx dy dz = \int_0^{2\pi} \int_0^2 \int_{r^2}^{4} r \cos \theta r dz dr d\theta.$$

13. Let D denote the solid bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2+y^2}$. Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2} dx dy dz$.

    Solution: Please see Figure 7.

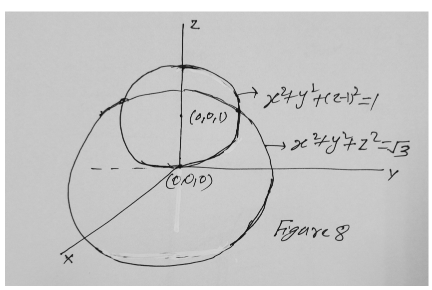


We use the spherical coordinates. The equation $z = \sqrt{x^2+y^2}$ changes to $\rho\cos\phi = \rho\sin\phi$. This implies that $\phi = \frac{\pi}{4}$. The equation $z = 4$ is written as $\rho\cos\phi = 4$. That is, $\rho = \frac{4}{\sec\phi}$. Therefore,

$$\iiint\limits_D \sqrt{x^2+y^2+z^2}dxdydz = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4\sec\phi} \rho\rho^2\sin\phi d\rho d\phi d\theta = 2\pi = 4^3\int_0^{\frac{\pi}{4}}\frac{\sin\phi}{\cos^4\phi}.$$

14. Parametrize the part of the sphere $x^2+y^2+z^2 = 16$, $-2 \le z \le 2$ using the spherical co-ordinates.
    Solution: By the spherical coordinates we can write the required surface as $S := r(\theta,\phi) = (4\sin\phi\cos\theta,\, 4\sin\phi\sin\theta,\, 4\cos\phi)$, where $0 \le \theta \le 2\pi$, $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$.
15. Let D denote the solid enclosed by the spheres $x^2+y^2+(z-1)^2 = 1$ and $x^2+y^2+z^2 = 3$. Using the spherical coordinates, set up iterated integral that gives the volume of D.

    Solution: Please Figure 8.



By solving $x^2+y^2+(z-1)^2 = 1$ and $x^2+y^2+z^2 = 3$ we get $z = \frac{3}{2}$. That is, $\rho\cos\phi = \frac{3}{2}$. the equation $x^2+y^2+(z-1)^2 = 1$ becomes $\rho = 2\cos\phi$ in the spherical coordinates.

The required volumes is the sum of the volume of the portion of the region $x^2+y^2+z^2 \le 3$ that lies inside the cone $\rho = \frac{\pi}{6}$ and the volume of the portion of the region $x^2+y^2+(z-1)^2 \le 1$ that lies inside the sphere $x^2+y^2+z^2 = 3$. Therefore the required volume is given by

$$\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{\sqrt{3}} \rho^2 \sin\phi d\rho d\phi d\theta + \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{2\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta.$$

16. Let S be the part of the sphere $x^2+y^2+z^2=4$ that lies above the cone $z=\sqrt{x^2+y^2}$. Parametrize S by considering it as a graph and again by using the spherical coordinates.
Solution: The surface S is bounded below by $z=\sqrt{2}$ and above by $z=2$. By spherical coordinates, we get $\sqrt{2}\le 2\cos\phi\le 2$. This implies that $0\le\phi\le\frac{\pi}{4}$. Hence $S:=r(\theta,\phi)=(2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi)$, where $0\le\theta\le 2\pi$, $0\le\phi\le\frac{\pi}{4}$.
17. Let S denote the part of the plane $2x+5y+z=10$ that lies inside the cylinder $x^2+y^2=9$. Find the area of S.
(a) By considering S as a part of the graph $z=f(x,y)$, where $f(x,y)=10-2x-5y$.
(b) By considering S as a parametric surface $r(u,v)=(u\cos v, u\sin v, 10-u(2\cos v+5\sin v))$, $0\le u\le 3$ and $0\le v\le 2\pi$.
Solution: (a) The projection D of the surface on the xy-plane is $\{(x,y): x^2+y^2=9\}$. The required area is $\iint\limits_D \sqrt{1+f_x^2+f_y^2}\ dxdy = \iint\limits_D \sqrt{30}\ dxdy = 9\sqrt{30}\pi$.
(b) The area is $\int_0^3\int_0^{2\pi}|r_u\times r_v|dudv = \int_0^3\int_0^{2\pi} u\sqrt{30}\ dudv$.
18. Find the area of the surface $x=uv, y=u+v, z=u-v$, where $(u,v)\in D=\{(s,t)\in\mathbb{R}^2: s^2+t^2\le 1\}$.
Solution: The surface is given by $r(u,v)=(uv,u+v,u-v)$ and hence $|r_u\times r_v| = \sqrt{4+2(u^2+v^2)}$. Therefore the required area is

$$\iint\limits_D \sqrt{4+2(u^2+v^2)}dudv = \int_0^{2\pi}\int_0^1 \sqrt{4+2r^2}rdrd\theta.$$

19. Find the area of the part of the surface $z=x^2+y^2$ that lies between the cylinders $x^2+y^2=4$ and $x^2+y^2=16$.
Solution: The given surface $z=x^2+y^2$ can be parameterized as $R(r,\theta)=(r\cos\theta, r\sin\theta, r^2)$, $r\ge 0$, and $0\le\theta\le 2\pi$. Hence $|R_r\times R_\theta| = r\sqrt{4r^2+1}$. Since the projection of the part of the surface on the xy-plane is the region between $x^2+y^2=4$ and $x^2+y^2=4$, we get $2\le r\le 4$. Therefore the required area is $\int_0^{2\pi}\int_2^4 r\sqrt{4r^2+1}drd\theta$.
20. Let S be the part of the cylinder $y^2+z^2=1$ that lies between the planes $x=0$ and $x=3$ in the first octant. Evaluate $\iint\limits_S (z+2xy)d\sigma$.
Solution: The surface is $r(x,\theta)=(x,\cos\theta,\sin\theta)$, $0\le x\le 3$ and $0\le\theta\le\frac{\pi}{3}$. This implies $|r_x\times r_\theta|=1$. Hence

$$\iint\limits_S (z+2xy)d\sigma = \int_0^{\frac{\pi}{2}}\int_0^3 (\sin\theta+2x\cos\theta)(1)dxd\theta = \int_0^{\frac{\pi}{2}}(3\sin\theta+9\cos\theta)d\theta.$$