MA15010H: Multi-variable Calculus

(Practice problem set 6: Change of variables, triple integral)

September - November, 2025

1. Consider the transformation $T : [0, 2\pi]\times[0, 1] \to \mathbb{R}^2$ given by $T(u, v) = (2v\cos u, v\sin u)$.
   (a) For a fixed $v_o \in [0, 1]$, describe the set $\{T(u, v_o) : \ u \in [0, 2\pi]\}$.
   (b) Describe the set $\{T(u, v) : \ [0, 2\pi] \times [0, 1]\}$.
2. Let R be the region in $\mathbb{R}^2$ bounded by the straight lines $y = x,\ y = 3x$ and $x + y = 4$. Consider the transformation $T(u, v) = (u-v, u+v)$. Find the set S satisfying $T(S) = R$.
3. Evaluate $\iint\limits_R x dxdy$ where R is the region $1 \le x(1 - y) \le 2$ and $1 \le xy \le 2$.
4. Evaluate
   (a) $\int\limits_0^{\frac{1}{\sqrt{2}}} \int\limits_{x=y}^{\sqrt{1-y^2}} (x + y)dxdy$.
   (b) $\int\limits_0^{1} \int\limits_{x=0}^{1-y} \sqrt{x + y}(y - 2x)^2 dxdy$.
   (c) $\int\limits_1^{2} \int\limits_{x=0}^{y} \frac{1}{(x^2+y^2)^{\frac{3}{2}}} dxdy$.
   (d) $\int\limits_0^{2} \int\limits_{y=0}^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dydx$.
5. Using change of variables $u = x + y$ and $v = x - y$, show that

$$\int\limits_0^1 \int\limits_{y=0}^{y=x} (x - y)dydx = \int\limits_0^1 \int\limits_{u=v}^{2-v} \frac{v}{2} dudv.$$

6. Find the volume of the solid in the first octant bounded below by the surface $z = \sqrt{x^2 + y^2}$ above by $x^2 + y^2 + z^2 = 8$ as well as the planes $y = 0$ and $y = x$.
7. Find the volume of the solid bounded by the surfaces $z = 3(x^2+y^2)$ and $z = 4-(x^2+y^2)$.
8. Let D denote the solid bounded by surfaces $y = x,\ y = x^2,\ z = x$ and $z = 0$. Evaluate $\iiint\limits_D y dxdydz$.
9. Let D denote the solid bounded below by the plane $z + y = 2$, above by the cylinder $z + y^2 = 4$ and on the sides $x = 0$ and $x = 2$. Evaluate $\iiint\limits_D x dxdydz$.
10. Let $D = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \le 1\}$ and $E = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \le 1\}$. Show that $\iiint\limits_D dxdydz = \iiint\limits_E 24 dudvdw$.
11. Let D be the solid that lies inside the cylinder $x^2 + y^2 = 1$, below the cone $z = \sqrt{4(x^2 + y^2)}$ and above the plane $z = 0$. Evaluate $\iiint\limits_D x^2 dxdydz$.

12. Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} xdzdydx$.
13. Let D denote the solid bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2 + y^2}$. Evaluate $\iiint_D \sqrt{x^2 + y^2 + z^2}dxdydz$.
14. Parametrize the part of the sphere $x^2 + y^2 + z^2 = 16$, $-2 \leq z \leq 2$ using the spherical co-ordinates.
15. Let D denote the solid enclosed by the spheres $x^2+y^2+(z-1)^2 = 1$ and $x^2+y^2+z^2 = 3$. Using the spherical coordinates, set up iterated integral that gives the volume of D.
16. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$. Parametrize S by considering it as a graph and again by using the spherical coordinates.
17. Let S denote the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$. Find the area of S.
    (a) By considering S as a part of the graph $z = f(x, y)$, where $f(x, y) = 10 - 2x - 5y$.
    (b) By considering S as a parametric surface $r(u, v) = (u \cos v, u \sin v, 10 - u(2 \cos v + 5 \sin v))$, $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$.
18. Find the area of the surface $x = uv, y = u + v, z = u - v$, where $(u, v) \in D = \{(s, t) \in \mathbb{R}^2 : s^2 + t^2 \leq 1\}$.
19. Find the area of the part of the surface $z = x^2 + y^2$ that lies between the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 16$.
20. Let S be the part of the cylinder $y^2 + z^2 = 1$ that lies between the planes $x = 0$ and $x = 3$ in the first octant. Evaluate $\iint_S (z + 2xy)d\sigma$.