MA15010H (CSE), 2025: Multivarable Calculus, Hint/Model solution

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1. (a) Whether the set $\{(x,y,z)\in\mathbb{R}^3 : |x|+2|y|+3|z|^2<1\}$ is bounded in $\mathbb{R}^3$? $\boxed{\mathbf{1}}$

   Solution: Let $A=\{(x,y,z)\in\mathbb{R}^3:\ |x|+2|y|+3|z|^2<1\}$. Then $|x|<1,\ |y|<\frac{1}{2},\ |z|^2<\frac{1}{3}$. Hence

   $$|x|^2+|y|^2+|z|^2<1+\frac{1}{4}+\frac{1}{3}=\frac{19}{12}=:r.$$

   Thus $A\subset B_r(0)$, and therefore A is bounded.

   (b) Whether there exists an unbounded sequence (x_n) in $\mathbb{R}$ such that $((x_n,\sin x_n^2))$ has convergent subsequence? $\boxed{\mathbf{1}}$

   Solution: The sequence $x_n=1,\frac{1}{2},2,\frac{1}{3},\ldots$ is unbounded, while the subsequence $x_{n_k}=\left(\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right)$ is convergent. Since $\sin t$ is continuous, $\left(x_{n_k},\sin(x_{n_k}^2)\right)$ is a convergent subsequence.

   (c) Does there exist a continuous function $f:\mathbb{R}\to\mathbb{R}^2$ such that $f(e^{-n^2})=(n,\frac{1}{n})$ for each $n\in\mathbb{N}$? $\boxed{\mathbf{1}}$

   Solution: No. A continuous function maps bounded sets to bounded sets, which would fail for such f since $f(e^{-n^2})=\left(n,\frac{1}{n}\right)$ is unbounded in the first component.

2. Show that the set $\{x\in\mathbb{R}^m : 2\le\|x\|<3\}$ is neither open nor closed set in $\mathbb{R}^m$. $\boxed{\mathbf{2}}$

   Solution: Let $A=\{x\in\mathbb{R}^m : 2\le\|x\|<3\}$. Note that the sequence $x_n=(3-\frac{1}{n})e_1\in A$, and converges to $3e_1\notin A$. Hence A is not closed. Also, no ball with center $2e_1$ and any radius is complectly contained in A. Hence A is not open.

3. If (x_n) is sequence in $\mathbb{R}^m$ such that the series $\sum\limits_{n=1}^{\infty} n^3\|x_n\|^2<\infty$. Show that the series $\sum\limits_{n=1}^{\infty}\|x_n\|^2$ is convergent. $\boxed{\mathbf{2}}$

   Solution: Since $\sum n^3\|x_n\|^2<\infty$, we have $\sum n^6\|x_n\|^4<\infty$. Then

   $$\sum\|x_n\|^2=\sum\frac{n^3\|x_n\|^2}{n^3}\le\Big(\sum\frac{1}{n^6}\Big)^{1/2}\Big(\sum n^6\|x_n\|^4\Big)^{1/2}<\infty$$

   by the Cauchy–Schwarz inequality.

4. Let function $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

   $$f(x,y)=\begin{cases}\dfrac{\sin^2(x-y)}{|x|+|y|} & \text{if } |x|+|y|\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

   Check the continuity of f at $(0,0)$. $\boxed{\mathbf{3}}$

Solution: For $(h,k)\neq(0,0)$,

$$|f(h,k)-f(0,0)| = \left|\frac{\sin^2(h-k)}{|h|+|k|}\right| \le \frac{|h-k|^2}{|h|+|k|} \le \frac{(|h|+|k|)^2}{|h|+|k|} = |h|+|k| \le \sqrt{2}\sqrt{h^2+k^2},$$

where we used $|\sin t|\le|t|$ and the Cauchy–Schwarz inequality. Hence f is continuous at $(0,0)$.

5. Let $f:\mathbb{R}^2\to\mathbb{R}$ be such that $f\circ g$ is differentiable for every function $g:\mathbb{R}\to\mathbb{R}^2$ with $g(0)=(0,0)$. Show that all the directional derivative of f exist $(0,0)$. $\boxed{\mathbf{2}}$

Solution: By assumption, for each such g the limit

$$\lim_{t\to0}\frac{f(g(t))-f(g(0))}{t}$$

exists. Taking $g(t)=tv$ with $\|v\|=1$ yields

$$\lim_{t\to0}\frac{f(tv)-f(0,0)}{t}=D_vf(0,0),$$

so every directional derivative $D_vf(0,0)$ exists.

6. Show that the function f defined by $f(x,y)=\dfrac{1}{1+x-y}$ is differentiable at $(0,0)$. $\boxed{\mathbf{3}}$

Solution: We have

$$f_x(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{\frac{1}{1+h}-1}{h}=\lim_{h\to0}\frac{1-(1+h)}{h(1+h)}=-1, \qquad f_y(0,0)=1.$$

Let

$$\begin{aligned}\epsilon(h,k) &= \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}\\ &= \frac{\frac{1}{1+h-k}-1+h-k}{\sqrt{h^2+k^2}} = \frac{(h-k)^2}{(1+h-k)\sqrt{h^2+k^2}}.\end{aligned}$$

Using $|h|+|k|\le\sqrt{2}\sqrt{h^2+k^2}$,

$$|e(h,k)| \le \frac{|h-k|\,|h-k|}{(1+h-k)\sqrt{h^2+k^2}} \le \frac{\sqrt{2}(|h|+|k|)}{1+h-k} \xrightarrow[(h,k)\to(0,0)]{} 0.$$

Hence f is differentiable at $(0,0)$.

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