DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
MA201: Mathematics-III

Mid-Semester Examination (Complex Analysis) Maximum Marks: 40
Date: September 15, 2024 Time: 09:00 A.M. to 12:00 Noon

Roll No. ...., Name ...., Tutorial Group ......

A. *Please ensure that your* ***roll no, name, and tutorial group*** *are correctly indicated on the answer sheet and question paper.*

B. ***An answer without proper justification will attract a zero mark.***

---

1. Prove or disprove the following statements:

 (a) There exists a non-constant entire function f such that $f(z)$ is real for all $z \in \mathbb{R}$, satisfying $f\left(\frac{1}{2n+1}\right) = f\left(\frac{1}{2n}\right)$ for all $n \in \mathbb{N}$.

 (b) If f is a non-constant entire function, then e^f has an essential singularity at $z = \infty$.

 (c) If $f : B(0,2) \to \mathbb{C}$ is an analytic function satisfying $f(0) = 1$, and $\left|f(e^{i\theta})\right| > 2$ for all $\theta \in [-\pi, \pi]$, then there exists $z_0 \in B(0,2)$ such that $f(z_0) = 0$. (Here $B(0,2) = \{z \in \mathbb{C} : |z| < 2\}$.)

 (d) If f is analytic on the domain $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$ satisfying $|f(z)| \leq \log \dfrac{1}{|z|}$ for all $z \in D$, then f has a removable singularity at $z = 0$.

 (e) $\displaystyle\int_{|z-1|=1} \left(\sum_{n=0}^{\infty} \frac{2^n z^n}{3^n + 5^n} \right) dz = 2\pi i.$ $\boxed{5 \times 2}$

2. (a) Let $C = \{3e^{it} : 0 \leq t \leq \frac{\pi}{2}\}$. Show that $\left| \displaystyle\int_C \frac{e^{iz}}{\bar{z}^2 + \bar{z} + 1}\, dz \right| \leq \frac{3\pi}{10}$.

 (b) For $R > 1$, let $\Gamma_R = \{z \in \mathbb{C} : |z| = R\}$. Show that
 $$\lim_{R \to \infty} \left| \int_{\Gamma_R} \frac{\mathrm{Log}(z^2)}{z^2}\, dz \right| = 0.$$

(c) Find the points on $[0, 2\pi] \times [0, 2\pi]$ where $\sin z$ attains its maximum modulus. Justify your answer. $\boxed{3+3+4}$

3. Classify the singularities (removable/pole/essential/non-isolated) of the following functions at the specified points. Find the order if the singularity is a pole.

(i) $\cos\left(\frac{z}{1+z}\right)$ at $z=-1$ (ii) $\left(\frac{\sin(e^z-1)}{z\sinh z}\right)^2$ at $z=0$

(iii) $\cot\left(\frac{1}{z}\right)$ at $z=0$ (iv) $\exp\left(\frac{\sin z - z}{z^3}\right)$ at $z=0$. $\boxed{4\times 1}$

4. (a) Find a conformal map that takes $\{z = x+iy \in \mathbb{C} : x>0,\ y>0\}$ onto $\{w = u+iv \in \mathbb{C} : u<v\}$.

(b) Find the image of the region $\{z = x+iy \in \mathbb{C} :\ xy>1,\ x>0,\ y>0\}$ under the transformation $w(z) = z^2$.

(c) Find the image of the circle $\{z \in \mathbb{C} : |z| = 3\}$ on the unit sphere under the stereographic projection. $\boxed{2+3+3}$

5. (a) Show that the equation $z + e^{-z} - 2 = 0$ has exactly one root in the right half-plane $\{z = x+iy \in \mathbb{C} : x>0, y \in \mathbb{R}\}$. (Hint: choose a contour as the boundary of a large semi-disc)

(b) Use the residue theorem to find the Cauchy principal value of
$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2}\, dx.$ $\boxed{3+5}$

PAPER ENDS