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MA201: Mathematics-III(2024): MidSem Solution/Hint

1. Prove or disprove the following statements: $\boxed{5 \times 2}$

(a) There exists a non-constant entire function f such that $f(z)$ is real for all $z \in \mathbb{R}$, satisfying $f\left(\frac{1}{2n+1}\right) = f\left(\frac{1}{2n}\right)$ for all $n \in \mathbb{N}$.

Answer: Since $f(\frac{1}{2n+1}) = f(\frac{1}{2n})$ for all $n \in \mathbb{N}$, by Rolle's theorem, there exists a sequence $c_n \in (\frac{1}{2n+1}, \frac{1}{2n})$ such that $c_n \to 0$ and hence $f'(c_n) = 0$. By uniqueness theorem, we get $f' \equiv 0$. Hence f is constant.

(b) If f is a non-constant entire function, then e^f has an essential singularity at $z = \infty$.

Answer: Note that $e^{f(1/z)}$ has singularity at $z = 0$. If not, then $e^{f(1/z)}$ is analytic at $z = 0$. Hence $\lim\limits_{|z|\to 0} e^{f(1/z)}$ exists as $e^{f(1/z)}$ is continuous at $z = 0$ as well. That is, $\lim\limits_{|z|\to\infty} e^{f(z)}$ exists, and hence $e^{f(z)}$ is bounded. By Liouville's theorem, it follows that f is constant. Here $z = \infty$ is not a removable singularity of $e^{f(z)}$. If $z = \infty$ is a removable singularity of $e^{f(z)}$, then $\lim\limits_{|z|\to\infty} e^{f(z)}$ exists, and hence $e^{f(z)}$ is a bounded function. By Liouville's theorem, f is constant. Also, $z = \infty$ is not a pole of $e^{f(z)}$. For this, if $z = \infty$ is a pole of $e^{f(z)}$, then $\lim\limits_{|z|\to\infty} e^{-f(z)} = 0$. This implies $e^{-f(z)}$ is bounded, and by Liouville's theorem, f is constant. Thus, $z = \infty$ is an essential singularity of $e^{f(z)}$.

(c) If $f : B(0, 2) \to \mathbb{C}$ is an analytic function satisfying $f(0) = 1$, and $\left|f(e^{i\theta})\right| > 2$ for all $\theta \in [-\pi, \pi]$, then there exists $z_0 \in B(0, 2)$ such that $f(z_0) = 0$. (Here $B(0, 2) = \{z \in \mathbb{C} : |z| < 2\}$.)

Answer: Suppose f has no zero in the open disc $B(0, 2)$, the $\frac{1}{f}$ will be analytic on $B(0, 2)$ and $\left|\frac{1}{f}(e^{i\theta})\right| \leq \frac{1}{2}$. However, $\left|\frac{1}{f}(0)\right| = 1$. This, in fact, contradicts the maximum modulus theorem. Thus, there is $z_0 \in B(0, 2)$ such that $f(z_0) = 0$.

Remark: $f(z_0) \neq 0$ for any $z_0 \in B(0, 2)$ does not mean that f will attain its minimum modulus at $z = 0$. The minimum and maximum will attended only on boundary $|z| = 2$.

(d) If f is analytic on the domain $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$ satisfying $|f(z)| \leq \log \dfrac{1}{|z|}$ for all $z \in D$, then f has a removable singularity at $z = 0$.

Answer: We claim that $\lim\limits_{z\to 0} zf(z) = 0$. Notice that $|zf(z) \leq \left|z \log \frac{1}{|z|}\right| = \left|\dfrac{\log t}{t}\right|$, if $|z| = \frac{1}{t}$. Then $\lim\limits_{r\to\pm\infty} \left|\dfrac{\log t}{t}\right| = 0$. Hence $z = 0$ is a removable singularity.

(e) $\displaystyle\int_{|z-1|=1} \left(\sum_{n=0}^{\infty} \frac{2^n z^n}{3^n + 5^n} \right) dz = 2\pi i.$

Answer: It can be shown that $f(z) = \sum_{n=0}^{\infty} \dfrac{2^n z^n}{3^n + 5^n}$ has radius of convergence $\frac{5}{2}$. This means $f : B(0, 5/2) \to \mathbb{C}$ is analytic. Hence, by Cauchy's theorem, we get $\displaystyle\int_{|z-1|=1} f(z)\, dz = 0.$

Remark: Note that summation and integration can commute if convergence is uniform on the curve.

2. (a) Let $C = \{3e^{it} : 0 \le t \le \frac{\pi}{2}\}$. Show that $\left| \displaystyle\int_C \frac{e^{iz}}{\bar{z}^2 + \bar{z} + 1} dz \right| \le \dfrac{3\pi}{10}$. $\boxed{3+3+4}$

Answer: By repeated use of triangle inequality, we can simplify the expression

$$|\bar{z}^2 + \bar{z} + 1| \ge ||\bar{z}^2| - |\bar{z} + 1|| \ge ||\bar{z}^2| - |\bar{z}| - 1| = 9 - 3 - 1 = 5.$$

Also, $|e^{iz}| = e^{-y} = e^{-3\sin t} < 1$. Hence, by ML-inequality, we get the required.

(b) For $R > 1$, let $\Gamma_R = \{z \in \mathbb{C} : |z| = R\}$. Show that $\displaystyle\lim_{R\to\infty} \left| \int_{\Gamma_R} \frac{\mathrm{Log}(z^2)}{z^2} dz \right| = 0.$

Answer: Let $z = R\,e^{it}$, where $t \in [-\pi, \pi]$. Then

$$\begin{aligned} \left| \int_{\Gamma_R} \frac{\mathrm{Log}(z^2)}{z^2} dz \right| &= \left| \int_{-\pi}^{\pi} \frac{\mathrm{Log}(R^2 e^{2it})}{R^2 e^{2it}} iRe^{it} dt \right| \\ &\le \int_{-\pi}^{\pi} \frac{|\ln(R^2) + 2it|}{R} dt \\ &\le \int_{-\pi}^{\pi} \frac{2|\ln R| + 2\pi}{R} dt \to 0 \text{ as } R \to \infty \end{aligned}$$

(c) Find the points on $[0, 2\pi] \times [0, 2\pi]$ where $\sin z$ attains its maximum modulus. Justify your answer.

Answer: Note that $|\sin z|^2 = \sinh^2 y + \sin^2 x$ on $[0, 2\pi]$. By the maximum modulus theorem, it is enough to analyze values of it on the boundary of the square.

$$|\sin z|^2 = \begin{cases} \sinh^2 2\pi & \text{if } x = 0 \text{ or } x = 2\pi \\ 1 & \text{if } y = 0 \\ \sinh^2 2\pi + 1 & \text{if } y = 2\pi. \end{cases}$$

Hence the maximum value of $|\sin z| = \sqrt{\sinh^2 2\pi + 1}$ that attended at $\frac{\pi}{2} + i2\pi$.

3. Classify the singularities (removable/pole/essential/non-isolated) of the following functions at the specified points. Find the order if the singularity is a pole. $\boxed{4 \times 1}$

(i) $\cos\left(\dfrac{z}{1+z}\right)$ at $z=-1$ (ii) $\left(\dfrac{\sin(e^z-1)}{z\sinh z}\right)^2$ at $z=0$

(iii) $\cot\left(\dfrac{1}{z}\right)$ at $z=0$ (iv) $\exp\left(\dfrac{\sin z - z}{z^3}\right)$ at $z=0$.

Answer: (i) Essential singularity, (ii) Pole of order 2, (iii) Non-isolated singularity (iv) Removable.

4. (a) Find a conformal map that takes $\{z = x+iy \in \mathbb{C} : x>0,\ y>0\}$ onto the region $\{w = u+iv \in \mathbb{C} : u<v\}$. $\boxed{2+3+3}$

   Answer: The required conformal transformation is $w(z) = e^{\frac{i\pi}{4}} z^2$.

   (b) Find the image of the region $\{z = x+iy \in \mathbb{C} :\ xy>1,\ x>0,\ y>0\}$ under the transformation $w(z) = z^2$.

   Answer: Let $w(z) = z^2 = u+iv$. Then $u(x,y) = x^2-y^2$ and $v(x,y) = 2xy$. The curve $xy=1$ is mapped under $w(z)$ to $\left(x^2 - \frac{1}{x^2}\right) + 2i$. Also, $w(2+2i) = 8i$. Hence, the image under $w(z)$ is $\{w = u+iv :\ u \in \mathbb{R},\ v>2\}$.

   (c) Find the image of the circle $\{z \in \mathbb{C} : |z| = 3\}$ on the unit sphere under the stereographic projection.

   Answer: Let $z = x+iy$, then the corresponding point on the unit sphere S^2 is given by

$$(X,Y,Z) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

   We need to find the image of the circle $|z|=3$. That is, $z = 3\cos t + 3\imath \sin t$, where $t \in [0,2\pi]$. Then $(X,Y,Z) = \left(\dfrac{3\cos t}{5}, \dfrac{3\sin t}{5}, \dfrac{4}{5}\right)$, which satisfying $X^2+Y^2 = \left(\frac{4}{5}\right)^2$, and $Z = \frac{4}{5}$. This a circle on the sphere at the height 4/5 from the equator.

5. (a) Show that the equation $z + e^{-z} - 2 = 0$ has exactly one root in the right half-plane $\{z = x+iy \in \mathbb{C} : x>0, y \in \mathbb{R}\}$. (Hint: choose a contour as the boundary of a large semi-disc) $\boxed{3+5}$

   Answer: Consider the curve $\Gamma_R = [-iR, iR] \cup \{Re^{i\theta} : \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}\}$ with sufficiently large R. Write $f(z) = z-2$ and $g(z) = e^{-z}$. Then $|g(z)| = 1$ on $[-iR, iR]$. Also, $|e^{-z}| = |e^{-R\cos\theta}| < 1$ on $\{z = Re^{i\theta} : \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$. Hence, $|g(z)| \le 1 < |z-2| = |f(z)|$, since $|z-2| = \sqrt{(R\cos\theta - 2)^2 + R^2\sin^2\theta} > 1$, whenever $R \ge 1$ and $|z-2| = |(it)^2 - 2| = \sqrt{4+t^2} > 1$ on $[-iR, iR]$. Since $f(z)$ has one root inside Γ_R. By Rouché's theorem, it follows that $f(z)+g(z) = z+e^{-z}-2$ has exactly one root inside Γ_R for all $R \ge$. Hence the given equation has exactly one root in the right half-plane.

   (b) Use the residue theorem to find the Cauchy principal value of
$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2}\,dx.$$

   Answer: Let $\Gamma_R = [-R, R] \cup \{Re^{i\theta} : 0 \le \theta \le \pi\}$, and $f(z) = \dfrac{z^3 e^{iz}}{(z^2+1)^2}$. By the residue theorem, we have $\displaystyle\int_{\Gamma_R} f(z)dz = 2\pi i \times \mathrm{Res}(f, i)$. By the residue formula, we get $\operatorname{Res}(f,i) = \dfrac{d}{dz}\left(\dfrac{z^3 e^{iz}}{(z^2+1)^2}\right)\Big|_{z=i} = \dfrac{1}{4e}$. Hence, $\displaystyle\int_{\Gamma_R} f(z)dz = 2\pi i \times \operatorname{Res}(f,i) = \frac{\pi i}{2e}$. Now, by Jordan's lemma, we get $\left|\displaystyle\int_0^{\pi} f(Re^{i\theta}) iRe^{i\theta} d\theta\right| \to 0$ as $R \to 0$. Therefore, $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \frac{\pi i}{2e}$. The Cauchy's principle value integral is $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2}\, dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+1)^2}\, dx\right) = \frac{\pi}{2e}$.