DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati
MA201: Mathematics-III: Hint/ model solutions

Quiz - I (Complex Analysis) Maximum Marks: 10
Date: 24 August 2024 Time: 10:00 A.M. to 11:00 A.M.

---

1. Prove that any non-constant harmonic function on a non-empty open set $D \subseteq \mathbb{C}$ is infinitely differentiable (partial derivatives of all orders exist and are continuous on D). **[3]**

   Answer: Let u be a harmonic function on $D \subseteq \mathbb{C}$. Since D is open, for $z_0 \in D$, there exists $r > 0$ such that $B(z_0, r) \subseteq D$. Then u is harmonic on the simply connected domain $B(z_0, r)$, there exists a harmonic conjugate v of u such that $f = u + iv$ is analytic on $B(z_0, r)$. By Cauchy's integral formula for higher derivative, f is infinitely differentiable. Hence, u is infinitely differentiable.

2. For $z = x + iy \in \mathbb{C}$, classify all entire functions $f(z) = u(x, y) + iv(x, y)$ that satisfy $u_y(x, y) = v_x(x, y)$. **[2]**

   Answer: Given that $f = u + iv$ satisfying $u_y = v_x$. We know that $f' = u_x + iv_x$ and Cauchy-Riemann equations are $u_x = v_y$ and $u_y = v_x$. Hence we get $u_y = 0 = v_x$. This implies $f' = u_x + i0$. Hence $f' = a$ (constant), since $\mathrm{Im} f'$ is constant. Define $g(z) = f(z) - az$. Then $g'(z) = f'(z) - a = 0$. Hence $g'(z) = b$. Thus, $f(z) = az + b$.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \begin{cases} z^2 \sin \frac{1}{z}, & z \neq 0; \\ 0, & z = 0. \end{cases}$ Discuss the continuity of the function f at $z = 0$. **[1]**

   Answer: For $x = 0$ and $y \to 0^+$, we get $\lim\limits_{y \to 0^+} -y^2 \sin \frac{1}{iy} = \lim\limits_{n \to \infty} \frac{1}{n^2} \sin(in) = \infty$. Hence, f is not continuous at $z = 0$.

4. Find all possible value(s) of $z \in \mathbb{C}$ satisfying the equation $2i = \frac{1 + e^{2z}}{\cos(iz)}$. **[1]**

   Answer: It follows from the given equation that $(e^{2z} + 1)(e^z - i) = 0$. If $e^z = i$, then
   $$z = \log i = \ln |i| + i\left(\frac{\pi}{2} + 2k\pi\right) = \frac{(4k+1)\pi}{2} i;\ k \in \mathbb{Z}.$$
   On the other hand if $e^{2z} + 1 = 0$, then
   $$z = \frac{1}{2} \log(-1) = \frac{1}{2} \ln |-1| + \frac{1}{2} i(\pi + 2k\pi) = \frac{(2k+1)\pi}{2} i;\ k \in \mathbb{Z}.$$

5. If g is an entire function satisfying $|g(z) - 2z| \leq 1$ on $|z| = 1$, show that $|g'(0)| \leq 3$. **[3]**

Answer: Define $f(z) = g(z) - 2z$. Then $f'(z) = g'(z) - 2$, and $f'(0) = g'(0) - 2$. By Cauchy integral formula, $|f'(0)| = \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^2} dz \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta \leq 1$.

Thus, $|g'(0) - 2| \leq 1$. That is, $|g'(0)| \leq 3$.

PAPER ENDS