MA547: Complex Analysis

(Assignment 1: Complex numbers system)

January - April, 2025

1. Show that $|z| \le |\mathrm{Re}(z)| + |\mathrm{Im}(z)| \le \sqrt{2}|z|$ for all $z \in \mathbb{C}$.
2. If $z_1, z_2 \in \mathbb{C}$, then $|z_1 + z_2| \le |z_1| + |z_2|$. Show that equality holds if and only if one of them is a nonnegative scalar multiple of the other.
3. If either $|z_1| = 1$ or $|z_2| = 1$, but not both, then prove that $\left|\dfrac{z_1 - z_2}{1 - \overline{z_1}\, z_2}\right| = 1$. What an exception must be made for the validity of the above equality when $|z_1| = |z_2| = 1$?
4. Show that the equation $z^4 + z + 5 = 0$ has no solution in the set $\{z \in \mathbb{C} \colon\ |z| < 1\}$.
5. If z and w are in $\mathbb{C}$ such that $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(w) > 0$, show that $|\frac{z-w}{z-\overline{w}}| < 1$.
6. When does $az + b\overline{z} + c = 0$ has exactly one solution?
7. Let $a, b, c \in \mathbb{C}$ such that $a \ne 0$ and $|a| \ne |c|$. Show that a root (in $\mathbb{C}$) of the equation $az^2 + bz + c = 0$ has modulus 1 iff $|\overline{a}b - \overline{b}c| = |a\overline{a} - c\overline{c}|$.
8. If $1 = z_0, z_1, \ldots, z_{n-1}$ are distinct n^{th} roots of unity, prove that
$$\prod_{j=1}^{n-1}(z - z_j) = \sum_{j=0}^{n-1} z^j.$$
9. Let $z, w \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$. Show that $|z + w|^2 \le (1 + \lambda)|z|^2 + (1 + \frac{1}{\lambda})|w|^2$.
10. Let $z, w \in \mathbb{C}$ such that $(1 + |z|^2)w = (1 + |w|^2)z$. Show that $z = w$ or $z\overline{w} = 1$.
11. Let $z \in \mathbb{C} \setminus \mathbb{R}$ such that $\dfrac{1 + z + z^2}{1 - z + z^2} \in \mathbb{R}$. Show that $|z| = 1$.
12. If $z, w \in \mathbb{D}$, then show that $|(1 - |z|^2)w + (1 - |w|^2)z| < |1 - z^2w^2|$.
13. If $z, w \in \mathbb{C}$, then show that $|1 + z| + |1 + w| + |1 + zw| \ge 2$.
14. Let $z \in \mathbb{C} \setminus \{1\}$ such that $z^n = 1$, where $n \in \mathbb{N}$. Show that $1 + 2z + \cdots + nz^{n-1} = \dfrac{n}{z-1}$.
15. Let $n \in \mathbb{N}$ and let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ such that $a_0 \ge a_1 \ge \cdots \ge a_{n-1} \ge a_n > 0$. Show that $|a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n| > 0$ for all $z \in \mathbb{D}$.
16. Let $z \in \mathbb{C} \setminus \{0\}$ such that $\left|z^3 + \dfrac{1}{z^3}\right| \le 2$. Show that $\left|z + \dfrac{1}{z}\right| \le 2$.
17. Show that all the roots of the equation $(z + 1)^3 + z^3 = 0$ lie on the line $\mathrm{Re}(z) + \frac{1}{2} = 0$.
18. Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that all the roots $z \in \mathbb{C}$ of the equation $\left(\dfrac{1 + iz}{1 - iz}\right)^n = a + ib$ are real iff $a^2 + b^2 = 1$.
19. Let $f(x) = \dfrac{1 + ix}{1 - ix}$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{C}$ is one-one. Also, determine the range of f.
20. Let $a \in \mathbb{R}$ such that $|a| < 1$ and let $f(z) = \dfrac{z - a}{1 - \overline{a}z}$ for all $z \in \mathbb{D}$. Show that $f : \mathbb{D} \to \mathbb{D}$ is one-one and onto.
21. Let $a, b \in \mathbb{C}$ and let $T(z) = az + b\overline{z}$ for all $z \in \mathbb{C}$. Show that $T : \mathbb{C} \to \mathbb{C}$ is one-one and onto iff $|a| \ne |b|$.

22. Let $z_1, z_2 \in \mathbb{C}$ such that $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$. Show that $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$.

23. Let $z \in \mathbb{C}$ such that $\mathrm{Re}(z^n) \geq 0$ for all $n \in \mathbb{N}$. Show that z is a non-negative real number.

24. If $d(z, w) = \dfrac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}$ for all $z, w \in \mathbb{C}$, then show that d is a metric on $\mathbb{C}$.