MA547: Complex Analysis

(Assignment 2: Topology of the complex plane, and continuity)

January - April, 2025

1. For each subset of $\mathbb{C}$, determine if it is open, closed, or not, with justification:
   (a) $\{z \in \mathbb{C} : Re(z) = 1 \text{ and } Im(z) \neq 4\}$
   (b) $B(1,1) \cup B(2, \frac{1}{2}) \cup B(3, \frac{1}{3})$
   (c) $\left\{z \in \mathbb{C} : \left|\frac{z-1}{z+1}\right| = 2\right\}$
   (d) $\{z \in \mathbb{C} : \sin(Re(z)) < Im(z) < 1\}$.
2. For each of the following subsets of $\mathbb{C}$, determine their interior, exterior, and boundary:
   (a) $\{z \in \mathbb{C} : |z| < 1 \text{ and } Im(z) \neq 0\} \cup \{z \in \mathbb{C} : |z| > 1 \text{ and } Im(z) = 0\}$
   (b) $\left\{r\left(\cos\left(\frac{1}{n}\right) + i \sin\left(\frac{1}{n}\right)\right) \in \mathbb{C} : r > 0, n \in \mathbb{N}\right\} \cup \{z \in \mathbb{C} : Re(z) < 0\}$.
3. Examine whether $\bigcup_{n=1}^{\infty} \{z \in \mathbb{C} : z^n = 1\}$ is open/closed in (a) $\mathbb{C}$ (b) $\partial\mathbb{D}$.
4. Determine all the limit points in $\mathbb{C}$ of the sets (a) $\left\{\frac{1}{m} + \frac{i}{n} : m, n \in \mathbb{N}\right\}$ (b) $\left\{x + \frac{i}{x} : x \in \mathbb{R},\ x > 0\right\}$.
5. State TRUE or FALSE with justification: Every uncountable subset of $\mathbb{C}$ has a limit point in $\mathbb{C}$.
6. Show that $\{\cos n + i \sin n : n \in \mathbb{N}\}$ is dense in $\partial\mathbb{D}$.
7. State TRUE or FALSE with justification: If (z_n) is a sequence in $\mathbb{C}$ such that (z_n) has no convergent subsequence, then it is necessary that $\lim_{n\to\infty} |z_n| = \infty$.
8. Let Ω be an open set in $\mathbb{C}$. Show that for each $n \in \mathbb{N}$, there exists a compact set K_n in $\mathbb{C}$ such that $\Omega = \bigcup_{n=1}^{\infty} K_n$.
9. Let (z_n) be a sequence in $\mathbb{C}$ and let $z \in \mathbb{C}$. Show that the series $\sum_{n=1}^{\infty} z_n$ converges with sum z iff both the series $\sum_{n=1}^{\infty} \text{Re}(z_n)$ and $\sum_{n=1}^{\infty} \text{Im}(z_n)$ converge with sums $\text{Re}(z)$ and $\text{Im}(z)$ respectively.
10. If $z \in \mathbb{C}$ such that $1 < |z| < 2$, then show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + z^2}$ is convergent.
11. If $z \in \mathbb{D}$, then show that the series $\sum_{n=1}^{\infty} \frac{z^{2n}}{2 + z^n + z^{5n}}$ is convergent.
12. Let (z_n) be a sequence in $\mathbb{C}$ such that $\text{Re}(z_n) \geq 0$ for all $n \in \mathbb{N}$. If both the series $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_n^2$ are convergent, then show that the series $\sum_{n=1}^{\infty} |z_n|^2$ is also convergent.
13. Let (z_n) be a sequence in $\mathbb{C}$ such that $\sup_{n\in\mathbb{N}} |\text{Arg}(z_n)| < \frac{\pi}{2}$. If the series $\sum_{n=1}^{\infty} z_n$ is convergent, then show that the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.

14. If $z \in \mathbb{C} \setminus \{0\}$ such that $|\mathrm{Arg}(z)| \leq \frac{\pi}{4}$, then show that the series $\sum_{n=1}^{\infty} \frac{z}{(1+z^2)^n}$ is convergent.
15. If $z \in \mathbb{C} \setminus \mathbb{N}$, then show that the series $\sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right)$ is convergent.
16. Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent series in $\mathbb{C}$. If $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N} \cup \{0\}$, then show that the series $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.
17. Examine whether the following subsets of $\mathbb{C}$ are connected.
    (a) $\{z \in \mathbb{C} : \mathrm{Re}(z),\ \mathrm{Im}(z) \in \mathbb{Q}\}$
    (b) $\{z \in \mathbb{C} : \mathrm{Re}(z)\mathrm{Im}(z) > 0\}$
    (c) $\{z \in \mathbb{C} : |z-1| \leq 1\} \cup \{z \in \mathbb{C} : |z-1+2i| < 1\}$
    (d) $\{z \in \mathbb{C} : \mathrm{Re}(z)^2 + \mathrm{Im}(z)^3 \in \mathbb{R} \setminus \mathbb{Q}\}$
    (e) $\{z \in \mathbb{C} : |z| \leq \sqrt{2}\} \cup \{z \in \mathbb{C} : |z-2-2i| \leq \sqrt{2}\}$
    (f) $\{z \in \mathbb{C} : |\mathrm{Re}(z)| < |\mathrm{Im}(z)|\}$
    (g) $\mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Re}(z) \in \mathbb{Q},\ \mathrm{Im}(z) \in \mathbb{Q}\}$
    (h) $\big\{x + i \sin \frac{1}{x} : x \in (0,1]\big\} \cup \big\{\frac{i}{3}\big\} \cup \big\{-\frac{i}{4}\big\}$
18. Show that the function $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = \begin{cases} z \sin \dfrac{1}{z} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$ is not continuous at 0.
19. If $f : \mathbb{C} \to \mathbb{C}$ is continuous such that $f(2z) = f(z)$ for all $z \in \mathbb{C}$, then show that f is a constant function.
20. State TRUE or FALSE with justification:
    (a) There exists a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(\sin n) = \frac{n\pi}{2}$ for all $n \in \mathbb{N}$.
    (b) If $f : \mathbb{C} \to \mathbb{C}$ is continuous and Ω is a bounded subset of $\mathbb{C}$, then $f(\Omega)$ must be a bounded subset of $\mathbb{C}$.
21. Let $f : K \to \mathbb{C}$ be a continuous function, where K is a compact set in $\mathbb{C}$. If $f(z) \neq 0$ for all $z \in K$, then show that there exists $r > 0$ such that $f(K) \subset \mathbb{C} \setminus B_r(0)$.
22. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous and $\lim_{|z| \to \infty} f(z) = 0$. Show that f is bounded and that there exists $z_0 \in \mathbb{C}$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \mathbb{C}$.
23. Let $f : \Omega \to \mathbb{C}$ be continuous, where Ω is a domain in $\mathbb{C}$. If $|f(z)^2 - 1| < 1$ for all $z \in \Omega$, then show that either $|f(z)-1| < 1$ for all $z \in \Omega$ or $|f(z)+1| < 1$ for all $z \in \Omega$.
24. Let Ω be a domain in $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be continuous such that $\{z \in \Omega : |f(z)| < |g(z)|\} \neq \emptyset$ and $\{z \in \Omega : |f(z)| > |g(z)|\} \neq \emptyset$. Show that $\{z \in \Omega : |f(z)| = |g(z)|\} \neq \emptyset$.

25. State TRUE or FALSE with justification: For every continuous function $f : \partial\mathbb{D} \to \mathbb{R}$, there exists $z \in \partial\mathbb{D}$ such that $f(z) = f(-z)$.
26. Show that there is no continuous function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $f(z)^2 = z$ for all $z \in \mathbb{C} \setminus \{0\}$.
27. State TRUE or FALSE with proper justification: An unbounded function $f : \mathbb{D} \to \mathbb{C}$ cannot be uniformly continuous (on $\mathbb{D}$).
28. Let $f(z) = \dfrac{z}{1 + |z|}$ for all $z \in \mathbb{C}$. Show that $f : \mathbb{C} \to \mathbb{D}$ is a homeomorphism.