MA547: Complex Analysis

(Assignment 3: Differentiability, analyticity and power series)

January - April, 2025

1. Show that $f : \mathbb{C} \to \mathbb{C}$ is nowhere differentiable in $\mathbb{C}$, where for each $z = x + iy \in \mathbb{C}$,
   (a) $f(z) = \mathrm{Re}(z)$
   (b) $f(z) = \mathrm{Im}(z)$
   (c) $f(z) = |z|$
   (d) $f(z) = 2x + ixy^2$
   (e) $f(z) = e^x(\cos y - i \sin y)$

2. Determine all the points of $\mathbb{C}$ at which $f : \mathbb{C} \to \mathbb{C}$ is differentiable, if for each $z = x + iy \in \mathbb{C}$,
   (a) $f(z) = x^3 + i(1 - y)^3$
   (b) $f(z) = z\mathrm{Im}(z)$
   (c) $f(z) = x^2 + iy^2$
   (d) $f(z) = x^2 + y + i(2y - x)$
   (e) $f(z) = z^2\overline{z}$
   (f) $f(z) = x^3y^2 + ix^2y^3$
   (g) $f(z) = |z|^4$
   (h) $f(z) = x^2 - y^2 + 2i|xy|$

3. Show that for each of the functions $f : \mathbb{C} \to \mathbb{C}$ defined as below, the Cauchy-Riemann equations are satisfied at 0 but $f'(0)$ does not exist (in $\mathbb{C}$).
   (a) $f(z) = \sqrt{|xy|}$ for all $z = x + iy \in \mathbb{C}$.
   (b) $f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$
   (c) $f(z) = \begin{cases} \dfrac{(1+i)\mathrm{Im}(z^2)}{|z|^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \begin{cases} \dfrac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} & \text{if } z = x + iy \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$
   Show that f is continuous. Also, show that the Cauchy-Riemann equations are satisfied at $(0, 0)$ but f is not differentiable at 0.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \begin{cases} \dfrac{x^3 y(y - ix)}{x^6 + y^2} & \text{if } z = x + iy \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$
Show that f is continuous. Also, show that the Cauchy-Riemann equations are satisfied at $(0, 0)$ but f is not differentiable at 0.

6. For each $z \in \mathbb{C}$, let $f(z) = \begin{cases} |z|^2 & \text{if } \mathrm{Re}(z), \mathrm{Im}(z) \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$
Determine all the points of $\mathbb{R}^2$ at which
   (a) at least one of the Cauchy-Riemann equations is satisfied.
   (b) both the Cauchy-Riemann equations are satisfied.

7. Show that $f : \mathbb{C} \to \mathbb{C}$ is not analytic at any point in $\mathbb{C}$, where for each $z = x + iy \in \mathbb{C}$,
   (a) $f(z) = z|z|$
   (b) $f(z) = \dfrac{z}{1 + |z|}$
   (c) $f(z) = xy + iy$
   (d) $f(z) = e^y(\cos x + i \sin x)$
   (e) $f(z) = x^2 + iy^3$

8. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there does not exist any point c on the line $y = 1 - x$ joining z_1 and z_2 such that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c)$$

(The mean value theorem does not extend to complex derivatives).

9. If $f(z)$ is a real valued function in a domain $D \subseteq \mathbb{C}$, then show that either $f'(z) = 0$ or $f'(z)$ does not exist in D.

10. Let $f : G \to \mathbb{C}$ be differentiable at a point $z_0 \in G$, where G is an open set in $\mathbb{C}$, and let $g(z) = \overline{f(z)}$ for all $z \in G$. Show that $g : G \to \mathbb{C}$ is differentiable at z_0 iff $f'(z_0) = 0$.

11. Let Ω_1 and Ω_2 be nonempty open sets in $\mathbb{C}$. Let $f : \Omega_1 \to \mathbb{C}$ be continuous such that $f(\Omega_1) \subset \Omega_2$ and let $g : \Omega_2 \to \mathbb{C}$ be holomorphic such that $g'(z) \neq 0$ for all $z \in \Omega_2$. If $g \circ f : \Omega_1 \to \mathbb{C}$ is holomorphic, then show that f is holomorphic.

12. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$, and let $g(z) = \overline{f(z)}$ for all $z \in \Omega$. If $g : \Omega \to \mathbb{C}$ is analytic, then show that f is a constant function.

13. Let G be an open set in $\mathbb{C}$ and let $G^* = \{\overline{z} : z \in G\}$. Show that G^* is open in $\mathbb{C}$.
If $f : G \to \mathbb{C}$ is analytic and $g(z) = \overline{f(\overline{z})}$ for all $z \in G^*$, then show that $g : G^* \to \mathbb{C}$ is analytic and that $g'(z) = \overline{f'(\overline{z})}$ for all $z \in G^*$.

14. Let Ω be a nonempty subset of $\mathbb{C}$ (considered to be $\mathbb{R}^2$ as a set). Show that a function $f : \Omega \to \mathbb{C}$ is differentiable as a function of two real variables at $z_0 \in \Omega^0$ iff there

exist $\varphi,\ \psi : \Omega \to \mathbb{C}$ such that both $\varphi,\ \psi$ are continuous at z_0 and $f(z) - f(z_0) = (z - z_0)\varphi(z) + (\overline{z} - \overline{z}_0)\psi(z)$ for all $z \in \Omega$.

15. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If L is a line in $\mathbb{C}$ and $f(\Omega) \subset L$, then show that f is a constant function.
16. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If P is the parabola in $\mathbb{C}$ given by the equation $y = x^2$ and $f(\Omega) \subset P$, then show that f is a constant function.
17. Let $f : \Omega \to \mathbb{C}$ be an analytic function, where Ω is a domain in $\mathbb{C}$. If $f' : \Omega \to \mathbb{C}$ is a constant function, then show that there exist $a, b \in \mathbb{C}$ such that $f(z) = az + b$ for all $z \in \Omega$.
18. Let $f = u + iv$ is an analytic function on the whole complex plane $\mathbb{C}$. If $u(x, y) = \phi(x)$ and $v(x, y) = \psi(y)$ prove that, for all $z \in \mathbb{C}$, $f(z) = az + b$ for some $a, b \in \mathbb{C}$.
19. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If for each $z \in \Omega$, either $f(z) = 0$ or $f'(z) = 0$, then show that f is a constant function.
20. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If for each $z \in \Omega$, $\mathrm{Re}(f(z)) = 0$ or $\mathrm{Im}(f(z)) = 0$, then show that $f : \Omega \to \mathbb{C}$ is a constant function.
21. For $z = x + iy \in \mathbb{C}$, classify all entire functions $f(z) = u(x, y) + iv(x, y)$ that satisfy $u_y(x, y) = v_x(x, y)$ for each $x, y \in \mathbb{R}$.
22. For $z = x + iy \in \mathbb{C}$, find all the entire functions $f(z) = u(x, y) + iv(x, y)$ satisfying $2u(x, y) + 3v(x, y) > 5$ for each $x, y \in \mathbb{R}$.
23. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If $z_0 \in \Omega$ such that $f'(z_0) \neq 0$, then show that there exists $\delta > 0$ such that f is one-one on $B_\delta(z_0)$.
24. Let u and v be nonconstant harmonic functions on $\mathbb{C}$.
    (a) If $U(x, y) = u(x, -y)$, is U also harmonic?
    (b) If v is a harmonic conjugate of u, is u a harmonic conjugate of v?
    (c) Is uv always harmonic? If not, produce an example.
25. If v is a harmonic conjugate of u (u, v real valued), prove that the functions uv and $u^2 - v^2$ are also harmonic.
26. What are all real valued harmonic functions u on D such that u^2 is also harmonic?
27. Find a harmonic conjugate, if it exists, of the following functions:
    (a) $u(x, y) = 2xy$.
    (b) $u(r, \theta) = r^n \cos n\theta$, $n \in \mathbb{N}$.
    (c) $u(x, y) = x^2 - y^2 + x + y - \frac{y}{x^2+y^2}$.
28. Let us define differential operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Let $f = u + iv$ be defined on an open set in $\mathbb{C}$. Show that:
    (a) f satisfies C-R equations if and only if $\frac{\partial}{\partial \overline{z}} f(z) = 0$.

(b) If $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then show that $\Delta f = 4\frac{\partial^2}{\partial z \partial \bar{z}} f$.

(c) Prove that the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}^n$ is harmonic for all $n \in \mathbb{N}$.

29. Let $u(x,y) = \begin{cases} \operatorname{Im}\left(\dfrac{1}{(x+iy)^2}\right) & \text{if } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Examine whether $u : \mathbb{R}^2 \to \mathbb{R}$ is harmonic.

30. Does there exist an analytic function $f : G \to \mathbb{C}$ for some open set G in $\mathbb{C}$ such that for all $z = x + iy \in G$,
(a) $\operatorname{Re}(f(z)) = x^2 - 2y$? (b) $\operatorname{Im}(f(z)) = x^3 - y^3$?

31. Determine all $v : \mathbb{R}^2 \to \mathbb{R}$ such that $f = u + iv : \mathbb{C} \to \mathbb{C}$ is analytic, where for all $(x,y) \in \mathbb{R}^2$,
(a) $u(x,y) = y^3 - 3x^2y$ (b) $u(x,y) = e^{-x}(x \sin y - y \cos y)$.
Also, express $f(z)$ in terms of $z \in \mathbb{C}$.

32. Determine all analytic functions $f = u + iv : \mathbb{C} \to \mathbb{C}$ such that $u(x,y) - v(x,y) = e^x(\cos y - \sin y)$ for all $(x,y) \in \mathbb{R}^2$.

33. Let $u : \Omega \to \mathbb{C}$ be a harmonic function, where Ω is a domain in $\mathbb{C}$. Show that $u_x - iu_y : \Omega \to \mathbb{C}$ is analytic.

34. Let $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$ be harmonic, where Ω is a domain in $\mathbb{C}$. Show that $(u_y - v_x) + i(u_x + v_y) : \Omega \to \mathbb{C}$ is analytic.

35. Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. Show that
(a) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\operatorname{Re}(f(z))^2 = 2|f'(z)|^2$
(b) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})|f(z)|^2 = 4|f'(z)|^2$
for all $z = x + iy \in \mathbb{C}$.

36. Determine the radius of convergence of each of the following power series. (a) $\sum_{n=1}^{\infty} n^{(-1)^n} z^n$ (b) $\sum_{n=0}^{\infty} z^{2^n}$ (c) $\sum_{n=0}^{\infty} (1 + \frac{1}{n})^{(-1)^n n^2} z^n$ (d) $\sum_{n=0}^{\infty} \left(\dfrac{2 + (-1)^n}{5 + (-1)^{n+1}}\right)^n z^n$
(e) $\sum_{n=0}^{\infty} a^{n^2} z^n$, where $a \in \mathbb{C}$.

37. For each $n \in \mathbb{N}$, let a_n be equal to the total number of (positive integer) divisors of n^{60}. Determine (with justification) the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$.

38. Determine all $z \in \mathbb{C}$ for which the following power series are convergent.
(a) $\sum_{n=1}^{\infty} \dfrac{2^n}{n^2}(z-2-i)^n$ (b) $\sum_{n=0}^{\infty} 2^n(z-2)^n$ (c) $\sum_{n=0}^{\infty} \dfrac{z^{4n}}{4n+1}$ (d) $\sum_{n=0}^{\infty} \dfrac{(2z-i)^n}{3n+1}$

39. Show that the radius of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} z^{n(n+1)}$ is 1. Also, examine the convergence of the power series for $z = 1, -1$, and i.

40. State TRUE or FALSE with justification: There exists a power series $\sum_{n=0}^{\infty} a_n(z-1+2i)^n$ in $\mathbb{C}$ which converges at $z = -3 + i$ and diverges at $z = -2 + 2i$.

41. Show that the radius of convergence R of a power series $\sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{C}$ is given by
$R = \sup\{|z| : z \in \mathbb{C},\ a_n z^n \to 0\} = \sup\{|z| : z \in \mathbb{C},\ \text{the sequence } (a_n z^n) \text{ is bounded}\}$.

42. Let R_1 and R_2 be the radii of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ respectively. If R is the radius of convergence of the power series $\sum_{n=0}^{\infty} (a_n + b_n) z^n$, then show that $R \geq \min\{R_1, R_2\}$.
Is it necessary that $R = \min\{R_1, R_2\}$? Justify.
If $R_1 \neq R_2$, then show that $R = \min\{R_1, R_2\}$.

43. Let R_1 and R_2 be the radii of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ respectively. If R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n z^n$, then show that $R \geq R_1 R_2$. (It is assumed that $R_1 R_2$ is defined.)
Is it necessary that $R = R_1 R_2$? Justify.

44. Let $a_n \in \mathbb{C} \setminus \{0\}$ for all $n \in \mathbb{N} \cup \{0\}$ and let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Is it necessary that the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1}{a_n} z^n$ is $\frac{1}{R}$? Justify.

45. Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Determine the radius of convergence of each of the following power series. (a) $\sum_{n=0}^{\infty} a_n^2 z^n$ (b) $\sum_{n=0}^{\infty} 2^n a_n z_n$ (c) $\sum_{n=1}^{\infty} n^n a_n z^n$

46. Examine whether $\left\{ \sum_{n=1}^{\infty} \frac{z^n}{n} : z \in \mathbb{C},\ \frac{1}{4} \leq |z| \leq \frac{3}{4} \right\}$ is a closed set in $\mathbb{C}$.

47. Examine whether the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$ is uniformly convergent in $\mathbb{C}$.

48. State TRUE or FALSE with justification: If $R > 0$ is the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, then the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ cannot converge uniformly on $\{z \in \mathbb{C} : |z - z_0| < R\}$.

49. Show that $|1 - (1 - z)e^z| \leq |z|^2$ for all $z \in \mathbb{D}$.

50. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \begin{cases} e^{-\frac{1}{z^4}} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$
Show that the Cauchy-Riemann equations for f are satisfied at every point of $\mathbb{R}^2$ but f is not continuous (and hence not differentiable) at 0.

51. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f'(z) = f(z)$ for all $z \in \mathbb{C}$ and $f(0) = 1$. Show that $f(z) = e^z$ for all $z \in \mathbb{C}$.

52. Let $f : \mathbb{C} \to \mathbb{C}$ be differentiable at 0 and $f'(0) = 1$. If $f(z + w) = f(z)f(w)$ for all $z, w \in \mathbb{C}$, then show that $f(z) = e^z$ for all $z \in \mathbb{C}$.

53. Let $f : \Omega \to \mathbb{C}$ be analytic, where Ω is a domain in $\mathbb{C}$. If $g(z) = e^{f(z)}$ for all $z \in \Omega$ and if $g : \Omega \to \mathbb{C}$ is a constant function, then show that f is a constant function.

    If f is assumed to be only continuous, then does a similar result hold? Justify.

54. For all $z \in \mathbb{C}$, show that

    (a) $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$.

    (b) $\cos(\pi + z) = -\cos z$ and $\sin(\frac{\pi}{2} + z) = \cos z$.

    (c) $\overline{\sin z} = \sin \overline{z}$ and $\overline{\cos z} = \cos \overline{z}$.

    (d) $\sin 2z = 2 \sin z \cos z$ and $\cos 2z = \cos^2 z - \sin^2 z$.

    (e) $\sin 3z = 3 \sin z - 4 \sin^3 z$ and $\cos 3z = 4 \cos^3 z - 3 \cos z$.

    (f) $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

    (g) $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

    (h) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.

    (i) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$.

55. For all $z, w \in \mathbb{C}$, show that

    (a) $\sin(z + w) = \sin z \cos w + \cos z \sin w$.

    (b) $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

56. Show that $\{\sin z : z \in \mathbb{C}\} = \mathbb{C}$ and $\{\cos z : z \in \mathbb{C}\} = \mathbb{C}$.

57. Let $f(z) = \sin z$ and $g(z) = \cos z$ for all $z \in \mathbb{C}$. Determine all the periods of $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$.

58. Show that $\{z \in \mathbb{C} : \cos z = 1\} = 2\pi\mathbb{Z}$ and $\{z \in \mathbb{C} : \sin z = 1\} = \dfrac{\pi}{2} + 2\pi\mathbb{Z}$.

59. Show that $\{z \in \mathbb{C} : \overline{\cos(iz)} = \cos(i\overline{z})\} = \mathbb{C}$ and $\{z \in \mathbb{C} : \overline{\sin(iz)} = \sin(i\overline{z})\} = \pi i\mathbb{Z}$.

60. State TRUE or FALSE with justification: If f is an entire function such that $|f(x)| \leq 10$ and $|f(iy)| \leq 10$ for all $x, y \in \mathbb{R}$, then there must exist $\lambda \in \mathbb{R}$ such that $|f(z)| \leq \lambda$ for all $z \in \mathbb{C}$.

61. If $z \in \mathbb{D}$, then show that the series $\sum\limits_{n=1}^{\infty} \sin(z^n)$ is absolutely convergent.

62. Let $f(z) = \begin{cases} |z|^2 \sin \dfrac{1}{|z|} & \text{if } z \in \mathbb{C} \setminus \{0\}, \\ 0 & \text{if } z = 0. \end{cases}$

    Examine whether $f : \mathbb{C} \to \mathbb{C}$ is differentiable at 0.

63. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \begin{cases} \dfrac{\sin z}{z} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$

    Show that f is an entire function.

64. Solve for $z \in \mathbb{C}$ the following equations.

    (a) $e^z = 2i$ (b) $\cos^2 z = 4$ (c) $\tan z = i$

65. Let $f(z) = \tan z$ for all $z \in \Omega = \mathbb{C} \setminus \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$. Determine the range of the function $f : \Omega \to \mathbb{C}$.

66. Show that $\log\left(\frac{1}{z}\right) = -\log z$ for all $z \in \mathbb{C} \setminus \{0\}$.

67. Examine whether (a) $\mathrm{Log}(1+i)^2 = 2\mathrm{Log}(1+i)$ (b) $\mathrm{Log}(-1+i)^2 = 2\mathrm{Log}(-1+i)$ (c) $\log(i^2) = 2\log i$.

68. Let Ω_1 and Ω_2 be domains in $\mathbb{C}$ such that $\Omega_1 \cap \Omega_2$ is connected. If there exists a branch of the logarithm on Ω_1 and there exists a branch of the logarithm on Ω_2, then is it necessary that there exists a branch of the logarithm on $\Omega_1 \cup \Omega_2$? Justify.

69. If $z_1 = 1+i$, $z_2 = 1-i$ and $z_3 = -1-i$, then examine whether $(z_1 z_2)^i = z_1^i z_2^i$ and $(z_2 z_3)^i = z_2^i z_3^i$, where only principal values are considered.

70. Show that all the values of $(1-i)^{\sqrt{2}i}$ lie on a straight line in the complex plane.