# MA547: Complex Analysis

(Assignment 4: Complex integration, Cauchy theorem, identity theorem and maximal principle)
January - April, 2025

1. Let $\gamma(0) = 0$ and $\gamma(t) = e^{\frac{i-1}{t}}$ for all $t \in (0, 1]$. Show that $\gamma : [0, 1] \to \mathbb{C}$ is a rectifiable path in $\mathbb{C}$. Also, determine the length of $\gamma$.
2. Let $\gamma(0) = 0$ and $\gamma(t) = t + it \sin \frac{1}{t}$ for all $t \in (0, 1]$. Show that $\gamma : [0, 1] \to \mathbb{C}$ is a path in $\mathbb{C}$ but $\gamma$ is not rectifiable.
3. Let $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [a, b] \to \mathbb{C}$ be rectifiable paths in $\mathbb{C}$ such that $\gamma_1(b) = \gamma_2(a)$. Show that the path $\gamma_1 + \gamma_2$ is rectifiable and that $L(\gamma_1 + \gamma_2) = L(\gamma_1) + L(\gamma_2)$.
4. If $\gamma$ is a rectifiable path in $\mathbb{C}$, then evaluate $\int\limits_{\gamma} |dz|$, with justification.
5. Let $\gamma$ be the polygon $[1 - i, 1 + i, -1 + i, -1 - i, 1 - i]$. Express $\gamma$ as a path and hence evaluate $\displaystyle\int_\gamma \frac{1}{z}\, dz$.
6. Evaluate the integral $\displaystyle\int_\gamma |z|\overline{z}dz$ where $\gamma$ is the circle $|z| = 2$.
7. Without evaluating the integral, show that
   (a) $\displaystyle\left|\int_\gamma \frac{z+4}{z^3-1}\, dz\right| \le \frac{6\pi}{7}$, where $\gamma(t) = 2e^{it}$ for all $t \in [0, \frac{\pi}{2}]$.
   (b) $\displaystyle\left|\int\limits_\gamma \frac{dz}{z^4}\right| \le 4\sqrt{2}$, where $\gamma$ denotes the line segment in $\mathbb{C}$ from $i$ to 1.
   (c) $\displaystyle\left|\int\limits_\gamma (e^z - \overline{z})\, dz\right| \le 60$, where $\gamma$ denotes the triangle $[0, 3i, -4, 0]$ in $\mathbb{C}$.
   (d) $\displaystyle\left|\int_\gamma \frac{dz}{z^2+1}\right| \le \frac{1}{2\sqrt{5}}$, where $\gamma$ is the straight line segment from 2 to $2 + i$.
8. State TRUE or FALSE with proper justification: If $f : \mathbb{D} \to \mathbb{C}$ is continuous such that $|f(z)| < 2$ for all $z \in \mathbb{D}$ and if $\gamma(t) = \frac{1}{2} + \frac{1}{4}e^{2it}$ for all $t \in [0, 2\pi]$, then it is necessary that $\left|\int\limits_\gamma f\right| < 2\pi$.
9. Let $f : \Omega \to \mathbb{C}$ be continuous, where $\Omega$ is a domain in $\mathbb{C}$. Let $z_0 \in \Omega$ and for each $r > 0$, let $\gamma_r(t) = z_0 + re^{it}$ for all $t \in [0, 2\pi]$. Show that $\displaystyle\lim_{r\to 0} \int_{\gamma_r} f(z)\, dz = 0$ and $\displaystyle\lim_{r\to 0} \int_{\gamma_r} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$.
10. Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded continuous function and for each $r > 0$, let $\gamma_r(t) = re^{it}$ for all $t \in [0, 2\pi]$. Show that $\displaystyle\lim_{r\to\infty} \int_{\gamma_r} \frac{f(z)}{(z - z_0)^2}\, dz = 0$ for all $z_0 \in \mathbb{C}$.
11. For each $r > 0$, let $\gamma_r(t) = re^{it}$ for all $t \in [0, \pi]$. Show that $\displaystyle\lim_{r\to\infty} \int_{\gamma_r} \frac{e^{iz}}{z}\, dz = 0$.
12. For each $r > 0$, let $\gamma_r(t) = re^{it}$ for all $t \in [0, 2\pi]$. If $p(z)$ and $q(z)$ are polynomials with $\deg q(z) \ge \deg p(z) + 2$, then show that $\displaystyle\lim_{r\to\infty} \int_{\gamma_r} \frac{p(z)}{q(z)}\, dz = 0$.

13. Let $f : G \to \mathbb{C}$ be continuous, where $G$ is an open set in $\mathbb{C}$. If $\gamma$ is a smooth path in $G$ such that $0 \notin \text{range}(\gamma)$, then show that

$$\left|\int_\gamma \frac{f(z)}{z} dz\right|^2 \leq L(\gamma) \left(\max_{z \in \text{range}(\gamma)} \frac{1}{|z|^2}\right) \int_\gamma |f(z)|^2 \, |dz|.$$

14. If $f(z) = \displaystyle\int_0^1 \frac{dt}{t-z}$ for all $z \in \mathbb{C} \setminus [0,1]$, then show that $f : \mathbb{C} \setminus [0,1] \to \mathbb{C}$ is continuous.

15. Let $f(z) = |z|^2$ for all $z \in \mathbb{C}$. Evaluate $\int\limits_{\gamma_1} f(z)\, dz$ and $\int\limits_{\gamma_2} f(z)\, dz$, where $\gamma_1 = [1, i]$ and $\gamma_2 = [1, 1+i, i]$.

    Hence show that $f : \mathbb{C} \to \mathbb{C}$ does not have any primitive on $\mathbb{C}$.

16. Let $z_0 \in \mathbb{C}$ and let $\gamma$ be a closed rectifiable path in $\mathbb{C}$ such that $z_0 \notin \text{range}(\gamma)$. If $n \in \mathbb{Z}$ and $n \neq 1$, then show that $\displaystyle\int_\gamma \frac{dz}{(z-z_0)^n} = 0$.

17. Let $f : G \to \mathbb{C}$ and $g : G \to \mathbb{C}$ be analytic, where $G$ is an open set in $\mathbb{C}$. If $\gamma$ is a rectifiable path in $G$ joining $z_1 \in G$ to $z_2 \in G$, then show that $\int\limits_\gamma fg' = f(z_2)g(z_2) - f(z_1)g(z_1) - \int\limits_\gamma f'g$.

18. Evaluate $\int\limits_\gamma z^2 \sin z \, dz$, where $\gamma(t) = e^{it}$ for all $t \in [0, \frac{\pi}{2}]$.

19. If $z, w \in \mathbb{C}$ such that $\text{Re}(z) \leq 0$ and $\text{Re}(w) \leq 0$, then show that $|e^z - e^w| \leq |z - w|$.

20. Let $f : \Omega \to \mathbb{C}$ be analytic, where $\Omega$ is a domain in $\mathbb{C}$. If $|f(z) - 1| < 1$ for all $z \in \Omega$ and if $\gamma$ is a closed rectifiable path in $\Omega$, then show that $\displaystyle\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$.

21. Examine whether the following functions have primitives.

    (a) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, defined by $f(z) = \dfrac{1}{z}$ for all $z \in \mathbb{C} \setminus \{0\}$.

    (b) $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = e^{-z^2}$ for all $z \in \mathbb{C}$.

    (c) $f : \mathbb{C} \setminus \{i, -i\} \to \mathbb{C}$, defined by $f(z) = \dfrac{1}{z^2+1}$ for all $z \in \mathbb{C} \setminus \{i, -i\}$.

    (d) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, defined by $f(z) = \dfrac{\sin z}{z^2}$ for all $z \in \mathbb{C} \setminus \{0\}$.

22. If $\gamma(t) = 1 + 2e^{it}$ for all $t \in [0, 2\pi]$, then explain clearly why Cauchy's theorem for star-shaped domain cannot be applied to get $\int\limits_\gamma \frac{dz}{z-1} = 0$.

23. If $\gamma(t) = e^{it}$ for all $t \in [0, 2\pi]$, then evaluate

    (a) $\displaystyle\int_\gamma \frac{\text{Re}(z)}{2z-1}\, dz$ (b) $\displaystyle\int_\gamma \frac{|dz|}{|z-a|^2}$, where $a \in \mathbb{D}$ (c) $\int\limits_\gamma |z-1|\, |dz|$ (d) $\displaystyle\int_\gamma \left(\frac{z-2}{2z-1}\right)^3 dz$

24. Let $r \in \mathbb{R} \setminus \{1, 2\}$ and $r > 0$. If $\gamma(t) = re^{it}$ for all $t \in [0, 2\pi]$, then determine all possible values of $\displaystyle\int_\gamma \frac{e^{\sin z^2}}{(z^2+1)(z-2i)^3}\, dz$.

25. Let $r \in \mathbb{R} \setminus \{2\}$ and $r > 0$. If $\gamma(t) = re^{it}$ for all $t \in [0, 2\pi]$, then determine all possible values of $\displaystyle\int_\gamma \frac{z^2+1}{z(z^2+4)}\, dz$.

26. Evaluate $\int\limits_0^{2\pi} e^{e^{2it} - 3it}\, dt$.

27. State TRUE or FALSE with justification: There exists a branch of the logarithm on $\mathbb{C} \setminus [-10, 10]$.
28. Let $f(z) = \dfrac{2z^3+1}{z^2+z}$ for all $z \in \mathbb{C} \setminus \{-1, 0\}$. Determine the Taylor series of $f : \mathbb{C} \setminus \{-1, 0\} \to \mathbb{C}$ about $i$.
29. Let $f : G \to \mathbb{C}$ be analytic, where $G$ is an open set in $\mathbb{C}$. Let $z_0 \in G$ and $g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \in G \setminus \{z_0\}, \\ f'(z_0) & \text{if } z = z_0. \end{cases}$ Examine whether $g : G \to \mathbb{C}$ is analytic.
30. State TRUE or FALSE with justification: If $f : \mathbb{C} \to \mathbb{C}$ is a non-constant analytic function, then there must exist a sequence $(z_n)$ in $\mathbb{C}$ such that $|z_n| > n$ and $|f(z_n)| > n$ for all $n \in \mathbb{N}$.
31. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. If there exist $M, r > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \leq M|z|^n$ for all $z \in \mathbb{C}$ with $|z| > r$, then show that $f$ is a polynomial of degree at most $n$.
32. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(0) = 0$ and $\lim\limits_{|z| \to \infty} f(z) = 0$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.
33. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(0) = 0$ and $\lim\limits_{|z| \to \infty} \mathrm{Re}(f(z)) = 0$. Show that $f(z) = 0$ for all $z \in \mathbb{C}$.
34. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\mathrm{Re}(f(z)) > 0$ for all $z \in \mathbb{C}$. Show that $f$ is a constant function.
35. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic such that $\big|\mathrm{Re}(f(z)) + \mathrm{Im}(f(z))\big| \leq 1$ for all $z \in \mathbb{C}$. Show that $f$ is a constant function.
36. State TRUE or FALSE with justification: There exists a non-constant bounded analytic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.
37. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(z+1) = f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Show that $f$ is a constant function.
38. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.
39. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that for each $z \in \mathbb{C}$, either $|f(z)| \leq 1$ or $|f'(z)| \leq 1$. Show that there exist $a, b \in \mathbb{C}$ such that $f(z) = az + b$ for all $z \in \mathbb{C}$.
40. If $f(z) = \sum\limits_{n=1}^{\infty} \dfrac{nz^n}{1 - z^n}$ for all $z \in \mathbb{D}$, then show that $f : \mathbb{D} \to \mathbb{C}$ is analytic.
41. Let $G$ be an open set in $\mathbb{C}$. For each $n \in \mathbb{N}$, let $f_n : G \to \mathbb{C}$ be analytic and let $f : G \to \mathbb{C}$. If $f_n \to f$ uniformly on each compact subset of $G$, then show that for each $k \in \mathbb{N}$, $f_n^{(k)} \to f^{(k)}$ uniformly on each compact subset of $G$.
42. Let $f : \Omega \to \mathbb{C}$ be an analytic function, where $\Omega$ is a domain in $\mathbb{C}$. If $\{z \in \Omega : f(z) = 0\}$ is uncountable, then show that $f(z) = 0$ for all $z \in \Omega$.

43. Let $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be analytic, where $\Omega$ is a domain in $\mathbb{C}$. If $f(z)g(z) = 0$ for all $z \in \Omega$, then show that $f(z) = 0$ for all $z \in \Omega$ or $g(z) = 0$ for all $z \in \Omega$.
44. Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ and $g : \overline{\mathbb{D}} \to \mathbb{C}$ be continuous such that $f$ and $g$ are analytic on $\mathbb{D}$. If $f(z) = g(z)$ for all $z \in \mathbb{C}$ with $|z| = 1$, then show that $f(z) = g(z)$ for all $z \in \mathbb{D}$.
45. Prove the fundamental theorem of algebra using the maximum modulus theorem.
46. Let $f$ be an entire function such that $|f(0)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Then either $f(0) = 0$ or $f$ is constant.
47. If $f : \mathbb{D} \to \mathbb{C}$ is analytic such that $|f'(z)| \leq k$ for all $z \in \mathbb{D}$, then $|f(z_1) - f(z_2)| \leq k|z_1 - z_2|$ for every pair of points $z_1$ and $z_2$ in $\mathbb{D}$.
48. If an entire function $f$ is such that $f(z)$ is real for all $z \in \mathbb{R}$, and $f\left(\frac{1}{2n+1}\right) = f\left(\frac{1}{2n}\right)$ for all $n \in \mathbb{N}$, then $f$ is a constant function.
49. Prove that any non-constant harmonic function on a non-empty open set $D \subsetneq \mathbb{C}$ is infinitely differentiable (partial derivatives of all orders exist and are continuous on $D$).
50. Let $f : \mathbb{C} \to \mathbb{C}$ be a function which is analytic on $\mathbb{C} \setminus \{0\}$ and bounded on $B(0, \frac{1}{2})$. Show that $\displaystyle\int_{|z|=R} f(z)dz = 0$ for all $R > 0$.
51. If $g$ is an entire function satisfying $|g(z) - 2z| \leq 1$ on $|z| = 1$, show that $|g'(0)| \leq 3$.
52. Suppose $f$ is analytic on the open unit disc D and it satisfies $|f(z)| \leq 1$ for all $z \in D$. Show that $|f'(0)| \leq 1$.
53. If $f : \mathbb{C} \to \mathbb{C}$ is continuous and analytic on $\mathbb{C} \setminus [-1, 1]$, then show that $f$ is entire.
54. Define $F(z) = \int\limits_0^1 \sin t^2\, e^{-itz} dt$. Show that $F$ is entire and satisfying $|F(z)| \leq A\, e^{B|y|}$ for $z = x + iy$ and for some positive constants $A$ and $B$.
55. Find all the entire functions $f$ such that $f(x) = e^x$ for all $x$ in $\mathbb{R}$.
56. If an entire function $f$ does not meet either real or imaginary axis, then show that $f$ is constant.
57. Let $f$ and $g$ be analytic functions on a domain $D$ in $\mathbb{C}$. If $\bar{f}\, g$ is analytic, then show that either $f$ is constant or $g \equiv 0$.
58. Let $f$ be a bounded analytic function on the right half plane (RHP). If $f$ is continuous on the imaginary axis and satisfies $\sup\limits_{y \in \mathbb{R}} |f(iy)| \leq M$, then show that $|f(z)| \leq M$ on the RHP. (Hint: Use maximum modulus theorem to $g_\epsilon(z) = (z+1)^{-\epsilon} f(z)$ on an appropriate semi-disc.)