MA547: Complex Analysis

(Assignment 5: Open mapping theorem, Schwarz lemma, Laurent series, singularity, residue theorem)
January - April, 2025

1. State TRUE or FALSE with justification:
   (a) There exists an entire function f such that $f(z)^2 = \sin z$ for all $z \in \mathbb{C}$.
2. Let $f : \mathbb{D} \to \mathbb{D}$ be a bijective holomorphic function such that $f(0) = 0$. If $f^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic, then show that there exists $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and $f(z) = \alpha z$ for all $z \in \mathbb{D}$.
3. Let $f : \mathbb{D} \to \mathbb{C}$ be a non-constant analytic function such that $f(0) = 1$. Show that there exist infinitely many $z \in \mathbb{D}$ such that $|f(z)| = 1$.
4. If $f : \mathbb{D} \to \mathbb{C}$ is analytic such that $\mathrm{Re} f'(z) < 0$ for all $z \in \mathbb{D}$, then f is injective on $\mathbb{D}$.
5. Let $f : G \to \mathbb{C}$ be continuous, where G is an open set in $\mathbb{C}$. If the function $f^2 : G \to \mathbb{C}$, defined by $f^2(z) = f(z)^2$ for all $z \in G$, is analytic, then show that f is analytic.
6. Let G be an open set in $\mathbb{C}$ and let $f : G \to \mathbb{C}$ be such that both the functions $f^2 : G \to \mathbb{C}$ and $f^3 : G \to \mathbb{C}$ are analytic, where $f^2(z) = f(z)^2$ and $f^3(z) = f(z)^3$ for all $z \in G$. Show that f is analytic.
7. Let f and g be entire functions such that $|f(z)| \le |g(z)|$ for all $z \in \mathbb{C}$. Show that there exists $\alpha \in \mathbb{C}$ such that $f(z) = \alpha g(z)$ for all $z \in \mathbb{C}$.
8. determine all entire function f such that $f(2z) = (f(z))^2$ for all $z \in \mathbb{C}$.
9. Determine all the singularities and their nature for
   (a) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, defined by $f(z) = \dfrac{\sin^2 z}{z^3}$ for all $z \in \mathbb{C} \setminus \{0\}$.
   (b) $f : \mathbb{C} \setminus S \to \mathbb{C}$, defined by $f(z) = \dfrac{e^z}{z(1 - e^{-z})}$ for all $z \in \mathbb{C} \setminus S$, where $S = \{z \in \mathbb{C} : e^{-z} \neq 1\}$.
   (c) $f : \mathbb{C} \setminus S \to \mathbb{C}$, defined by $f(z) = \tan \frac{1}{z}$ for all $z \in \mathbb{C} \setminus S$, where $S = \{z \in \mathbb{C} \setminus \{0\} : \cos \frac{1}{z} \neq 0\}$.
10. State TRUE or FALSE with justification: There exists a non-constant bounded analytic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.
11. Find the order of zero at $z = 0$ for $f : \mathbb{C} \to \mathbb{C}$, where for each $z \in \mathbb{C}$,
    (a) $f(z) = z^2(e^{z^2} - 1)$ (b) $f(z) = 6 \sin z^3 + z^3(z^6 - 6)$.
12. Find all the zeros together with their orders for
    (a) $f : \mathbb{C} \to \mathbb{C}$, defined by $f(z) = (1 - e^z)(z^2 - 4)^3$ for all $z \in \mathbb{C}$.
    (b) $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, defined by $f(z) = \dfrac{z^2 + 9}{z^4}$ for all $z \in \mathbb{C} \setminus \{0\}$.
13. Let $f : G \to \mathbb{C}$ and $g : G \to \mathbb{C}$ be analytic, where G is an open set in $\mathbb{C}$. If $z_0 \in G$ is a zero of both f and g of orders m and n respectively, then show that z_0 is a zero of $fg : G \to \mathbb{C}$ of order $m + n$.

14. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be analytic such that $|f(z)| \leq 1$ for all $z \in \mathbb{C} \setminus \{0\}$ with $|z| < 1$. If $r > 0$ and $\gamma(t) = re^{it}$ for all $t \in [0, 2\pi]$, then determine (with justification) $\int\limits_\gamma f$.

15. If $f(z) = \dfrac{1}{z(z-1)(z-2)}$ for all $z \in \mathbb{C} \setminus \{0, 1, 2\}$, then determine the Laurent series expansion of f in (a) $\operatorname{ann}(0; 0, 1)$, (b) $\operatorname{ann}(0; 1, 2)$, and (c) $\operatorname{ann}(0; 2, \infty)$.

16. If $f(z) = \dfrac{1}{z(z^2-1)}$ for all $z \in \mathbb{C} \setminus \{0, 1, -1\}$, then find the residue of $f : \mathbb{C} \setminus \{0, 1, -1\} \to \mathbb{C}$ at 1 using Laurent series expansion of f.

17. Let f be an entire function such that $f(n) = 0$ for all $n \in \mathbb{Z}$. If $g(z) = \dfrac{f(z)}{\sin(\pi z)}$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$, then show that each $n \in \mathbb{Z}$ is a removable singularity of $g : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$.

18. Let G be an open set in $\mathbb{C}$ and let $z_0 \in G$ be a zero of order $m \in \mathbb{N}$ of an analytic function $f : G \to \mathbb{C}$. Determine $\operatorname{Res}\left(\dfrac{f'}{f}; z_0\right)$.

19. If $u : \mathbb{R}^2 \to \mathbb{R}$ is a bounded harmonic function, then show that u is a constant function.

20. Evaluate $\int\limits_\gamma \tan z \, dz$, where $\gamma(t) = 2e^{it}$ for all $t \in [0, 2\pi]$.

21. Determine all possible values of $\displaystyle\int_\gamma \frac{dz}{z(z^2+1)}$, where γ is any closed rectifiable path in $\mathbb{C} \setminus \{0, i, -i\}$.

22. Let $f : G \to \mathbb{C}$ be analytic and one-one, where G is an open set in $\mathbb{C}$. Show that $f'(z) \neq 0$ for all $z \in G$.

23. Show that all the roots of the equation $z^7 - 5z^3 + 12 = 0$ lie in $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

24. If $\lambda \in \mathbb{R}$ such that $\lambda > 1$, then show that the equation $ze^{\lambda - z} = 1$ has exactly one root in $\mathbb{D}$ and that this root is real and positive.

25. If $a \in \mathbb{R}$ such that $a > 1$, then show that $\displaystyle\int_0^\pi \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2-1}}$.

26. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} \, dx = \frac{\pi}{\sqrt{2}}$.

27. Show that $\displaystyle\int\limits_\gamma \frac{e^{az}}{z^2+1} dz = 2\pi i \sin a$, where $\gamma(t) = 2e^{it}, \ t \in [0, 2\pi]$.