Complex number System

Understanding the geometry of complex function present a distinctive challenge when compared with the Euclidean plane. These complexities primarily rooted in the algebraic structure of the complex plane, which is significantly shaped by its specialized multi-plication properties. A pivotal element of this structure is captured by Cauchy - Riemann equs, which provide a framework different from that of the Frechet derivative.

To effectively navigate the complexities inherent in complex geometry, we need to investigate mappings, ~~transforms~~, transformations, and function rep's. Fundamental concepts such as analytic

Information. Conformal mapping, and holomorphic functions form the cornerstone of complex analysis. These principles facilitate a deeper understanding of complex functions and illustrate their geometric interpretations and applications across diverse field of mathematics and engineering.

Let us consider the equation

$$x^2+1 = 0 \quad \text{---}(*)$$

Then (*) has no real solution.
Let i (iota) be the solution of (*),
then $i^2=-1 \Rightarrow i=\sqrt{-1}$. Thus, i is not a real number any. we call it a imaginary number.

We write $z = x+iy$, $x,y\in\mathbb{R}$.
The number z is called complex number

The complex number z can also be represented as (3)

$$z \doteq (x, y) \;;\; x, y \in \mathbb{R}$$

note that $Z \longleftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$

For $z = x+iy$, the real part of z is denoted by $Re(z)$ and y = imaginary part of z is denoted by $Im(z)$.

If we write $i = (0,1)$, then for $Z = x+iy$, we identify x by $(x, 0)$ and y by $(0, y)$.

Note that $Re\, z = Im(iZ)$, $Im\, Z = -Re(iZ)$.

The set of all complex numbers is denoted by $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$ and is known as complex field.

For $Z_1 = x_1+iy_1$, $Z_2 = x_2+iy_2$, we write

$$Z_1 \cdot Z_2 = (x_1+iy_1)\cdot(x_2+iy_2)$$
$$= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Obviously, $i^2 = -1$. (4)

The multiplication operation can be justified by matrix def.^{n} too.

$$\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 & -(x_1y_2 + x_2y_1) \\ x_1y_2 + x_2y_1 & x_1x_2 - y_1y_2 \end{pmatrix}$$

If $z = x + iy \neq 0$, then $\det\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x^2 + y^2 \neq 0$.

then $z^{-1} \sim \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} = \frac{1}{x^2+y^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$

That is, $\frac{1}{z} = \frac{x - iy}{x^2+y^2}$ ——— (**)

Now, $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix}\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}^{-1}$.

The reflection of point $z = (x, y)$ about x-axis is known as complex conjugate.

That is, $\bar{z} := x - iy$. Thus $z^{-1} = \frac{\bar{z}}{|z|^2}$,

where $|z| = \sqrt{x^2+y^2} = \sqrt{\det\begin{pmatrix} x & -y \\ y & x \end{pmatrix}}$.

We can see that the complex plane $\mathbb{C}$ is closed under addition, multiplication, and division. That implies, $\mathbb{C}$ has multiplicative inverse. Thus, $\mathbb{C}$ is a field.

Note that (i) $Re(z) = \frac{1}{2}(z + \bar{z})$

and $Im\, z = \frac{1}{2i}(z - \bar{z})$

(ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(iv) $z z^{-1} \cong \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cong$.

(V) $\bar{\bar{z}} = z$, $Re\, z = Re\, \bar{z}$, $Im\, z = -Im\, \bar{z}$.

Modulus: For $z = x + iy$, the number $|z| := \sqrt{x^2+y^2}$ is known as modulus

Pf 2. We can see that

$$z\bar{z} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} x^2+y^2 & 0 \\ 0 & x^2+y^2 \end{pmatrix}$$

$$= x^2+y^2.$$

It is easy to see that

$$|x| = |\mathrm{Re}\, z| \le |z| \ \& \ |y| \le |\mathrm{Im}\, z| \le |z|$$

Also, $|\bar{z}| = |z|$, $|z_1 z_2| = |z_1||z_2|$ etc.

If $z_1, z_2 \in \mathbb{C}$, then

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$$

$$= |z_1|^2+|z_2|^2+z_1\bar{z}_2+\bar{z}_1 z_2$$

That is,

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2\,\mathrm{Re}\, z_1\bar{z}_2$$

$$\left(\because \mathrm{Re}(z_1\bar{z}_2) = \mathrm{Re}(\overline{z_1\bar{z}_2}) = \mathrm{Re}(\bar{z}_1 z_2)\right).$$

$$\le |z_1|^2+|z_2|^2+2|z_1\bar{z}_2|$$

$$= (|z_1|+|z_2|)^2$$

$$\Rightarrow |z_1+z_2| \le |z_1|+|z_2|. \ \text{(Triangle inequality)}$$

Now, $|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$ (7)

$\Rightarrow |z_1| - |z_2| \leq |z_1 - z_2|$

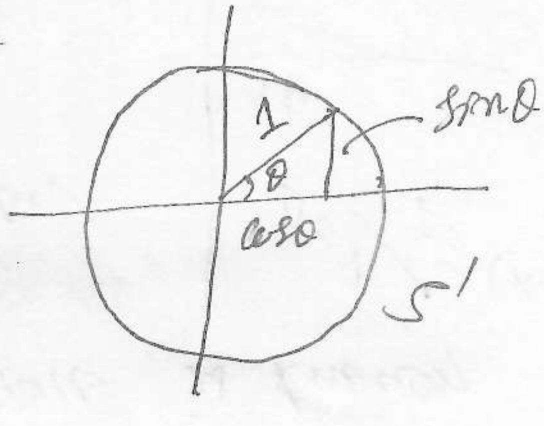
By replacing z_1 with z_2, we get

$$\big| |z_1| - |z_2| \big| \leq |z_1 - z_2|.$$

(that is, process of taking modulus is uniformly continuous).

<u>Polar rep^n of complex number:</u>

Consider the unit circle in the complex plane. Then any point on the unit circle can be rep^ed by



$\cos\theta + i\sin\theta$; $\theta \in [0, 2\pi)$

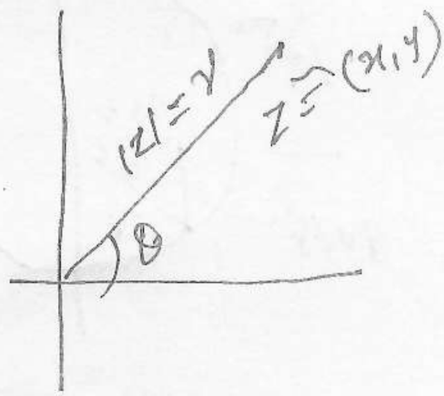
let $z \neq 0$, then $\left|\frac{z}{|z|}\right| = 1 \Rightarrow \frac{z}{|z|}$ is a point on the unit circle S^1, and hence

$\frac{z}{|z|} = \cos\theta + i\sin\theta$ for some $\theta \in [0, 2\pi)$.

That is, if we write $|z| = r > 0$, then

$$z = r(\cos\theta + i\sin\theta) \qquad (8)$$
$$= re^{i\theta};$$

where $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler formula.



Thus, any complex number $z \neq 0$, can be rep^d (coincidented) with its magnitude $|z| = \sqrt{x^2 + y^2}$ and an angle (direction) θ from the x-axis. Thus,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad |z| = r,$$

and $\theta = \tan^{-1}(y/x) \in [0, 2\pi)$.

If $z \neq 0$, the argument of z, denoted by $\arg(z) = \{\theta : z = re^{i\theta}\}$

& $\arg(z) = \{ \theta + 2k\pi \, ; \, z = re^{i\theta}, \, k \in \mathbb{Z} \}$

$\Rightarrow \arg(z)$ is a multi-valued function.

Denote $\arg(z) = \text{Arg}(z) + 2k\pi$, where

$\text{Arg}(z) \in (-\pi, \pi]$. (9)

Example: $\arg(i) = 2k\pi + \pi/2 \, ; \, k \in \mathbb{Z}$,

when $\text{Arg}(i) = \pi/2$.

Here, $\text{Arg}(z)$ is known as principle value of $\arg(z)$.

If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \quad \text{———} \quad (*)$$

If $z_1 \neq 0$ and $z_2 \neq 0$, then it follows from $(*)$ that

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

LHS $= \theta_1 + \theta_2 + 2k\pi$ and

RHS $= (\theta_1 + 2k_1\pi) + (\theta_2 + 2k_2\pi)$

$= \theta_1 + \theta_2 + 2k'\pi$; when $k' = k_1 + k_2 \in \mathbb{Z}$.

Note that $|e^{i(\theta_1+\theta_2)}| = 1$, $\forall\ \theta_1, \theta_2 \in \mathbb{R}$,

hence $|z_1 z_2| = |z_1||z_2|$.

If $z_j : j = 1, 2, \ldots n$; $z_j = r_j e^{i\theta_j}$. Then

$$z_1 \cdots z_n = r_1 \cdots r_n\, e^{i(\theta_1 + \cdots + \theta_n)} \text{ (by induction)}$$

If $z_j = z = r e^{i\theta}$, then

$$z^n = r^n e^{in\theta}$$

or $z^n = (r(\cos\theta + i\sin\theta))^n$ ——(***)

$$= r^n(\cos n\theta + i\sin n\theta) \text{ (by induction)}$$

Note that (***) can be obtained by without using Euler formula.

and hence

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

known as de Moivre's formula.

Now, for a given complex number $a \neq 0$, the question is how many z we

Can find such that $z^n = a$?

Let $a = |a| (\cos\alpha + i\sin\alpha)$. Then $|z| = |a|^{1/n}$

$\Rightarrow z = |a|^{1/n} \left(\cos\frac{\alpha}{n} + i\sin\frac{\alpha}{n}\right)$. (1)

However, there is more solutions and all of them can be listed as

$$|a|^{1/n} \left(\cos\left(\frac{\alpha + 2k\pi}{n}\right) + i \sin\left(\frac{\alpha + 2k\pi}{n}\right) \right)$$

$k = 0, 1, 2, \ldots n-1$.

Ex. Find the nth roots of the unity.

$1 = \cos 0 + i\sin 0 \Rightarrow$

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, \ldots n-1$$

are n roots of the unity.

Topology of the Complex plane:

Here we discuss a limited amount of informations related to open set, closed sets, interior, closure, boundary of sets in $\mathbb{C}$. At the end we indicate some facts related to connected sets. These definitions required to define and analyse limit, continuity, differentiability, integration, etc.

For $z_0 \in \mathbb{C}$, and $r > 0$, we write

$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$, known as open disc of radius r which is centered at z_0.

* The set $B(z_0, r) \setminus \{z_0\}$ is known as deleted nbd of z_0.

* A point $z_0 \in \mathbb{C}$ is called interior point

of a set $S \subset \mathbb{C}$ if $\exists\, r>0$ s.t.

$B(z_0, r) \subset S$. The set of interior points of S is denoted by S^0 or Int S.

<u>Notation:</u> $B(z_0, r) = B_r(z_0)$.

Example: $S = \{(x, \sin\frac{1}{x}) : x \neq 0,\ x \in \mathbb{R}\}$ has empty interior, i.e. $S^0 = \Phi$.

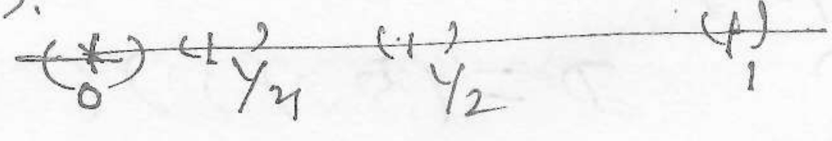
* A point $z_0 \in \mathbb{C}$ is said to be a boundary point of $S \subset \mathbb{C}$ if every nbd of z_0 intersect S and S^c.

That is, $B(z_0, r) \cap S \neq \Phi \neq B(z_0, r) \cap S^c$ $\forall\, r > 0$.

The boundary set of a set $S \subset \mathbb{C}$ is denoted by Bdy S or ∂S.

For $S = \{\frac{1}{n} : n = 1, 2, 3, \dots\}$

$\partial S = S \cup \{0\}$.



Ex. $\partial Q = \mathbb{R}$, $\partial(Q + iQ) = \mathbb{C}$ (14)

Note that interior is the largest open set contained in the set.

$S = \{(x, \frac{1}{x}) : x \neq 0, x \in \mathbb{R}\}$.

then S contain no ball inside it, hence $S^{\circ} = \Phi$.

(whereas the boundary is the largest (closed) set whose points and intersect both set and its complement).

* Exterior point of a set S are those points in $\mathbb{C}$ which are neither interior nor boundary point.
  It is denoted by $Ext(S)$.
  By definition, it is clear that
  $$Ext(S) = (S^{\circ} \cup \partial S)^{c}.$$
  Ex. $S = \{z \in \mathbb{C} : 1 < |z| \leq 2\}$, then

$Ext(S) = (S^\circ \cup \partial S)^c$ (15)

$= \{z \in \mathbb{C} : 1 \le |z| \le 2\}^c$

$= \{z \in \mathbb{C} : |z| > 1 \text{ and } |z| > 2\}.$

Ex. Show that $S \subseteq S^\circ \cup \partial S$.

Let $z \in S$ & $z \notin S^\circ$; then ~~for~~ $\forall r > 0$,

$B_r(z) \cap S \neq \Phi$ & $B_r(z) \cap S^c \neq \Phi$

$\Rightarrow z \in \partial S$.

Similarly, if $z \in S$ and $z \notin \partial S$

$\Rightarrow \exists\, r > 0$ st $B(z, r) \subset S$ & $B(z, r) \cap S^c$

$\Rightarrow z \in S^\circ$.

Thus, $S \subseteq S^\circ \cup \partial S$.

Ex. Show that $Ext(S)$ is an open set.

Let $z \in (S^\circ \cup \partial S)^c \subset S^c$

$\Rightarrow z \notin S$.

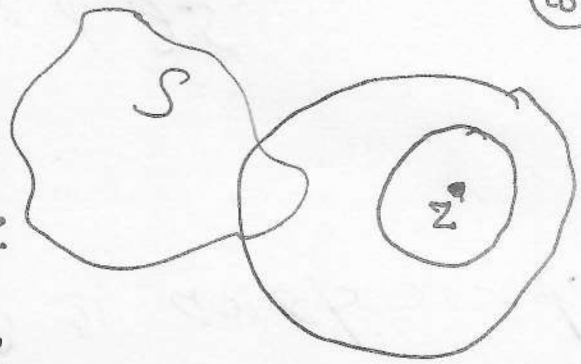
Since $z \notin S^\circ$ and $z \notin \partial S$

$\exists\, r > 0$ st $B_r(z) \cap S = \Phi$ (??)

$\Rightarrow B_r(z) \cap S^\circ = \Phi$ and $B_r(z) \subset S^c$.

Let $r = \text{dist}(z, S)$. 16

Then $B_{r/2}(z) \cap \partial S = \Phi$



$\Rightarrow B_{r/2}(z) \subset (S^\circ \cup \partial S)^c$

Hence, $\text{Ext}(S)$ is open.

* A set $S \subset \mathbb{C}$ is said to be bounded if $\exists\ K > 0$ s.t. $|z| \leq K$ for each $z \in S$.

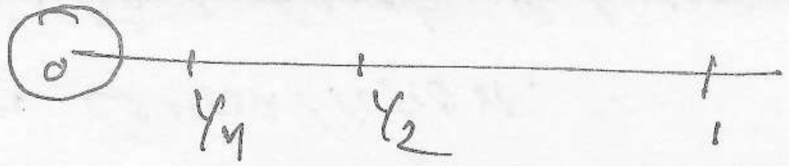
That is, $S \subset B_K(0)$.

Ex. $\{(x, \sin\frac{1}{x}) : x \in [-1, 1] \setminus \{0\}\}$ is a bounded set, whereas $\{(x, \sin\frac{1}{x}) : x \neq 0, x \in \mathbb{R}\}$ is unbounded.

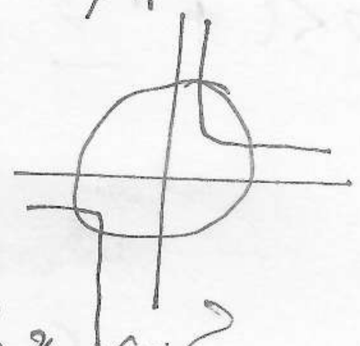
Limit point: A point $z \in \mathbb{C}$ is said to be limit point of

a set S if every deleted nbd of z contains a point of S.
The set of all limit points is denoted by S'.

Ex. $S = \{\frac{1}{n} : n = 1, 2, \dots\}$. Then $S' = \{0\}$.



Ex. $S = \{(x, \frac{1}{x}) : x \neq 0,\ x \in \mathbb{R}\}$.
Then $S' = S \cup \{(0,0)\}$



Ex. Let $S = \{(x, \sin\frac{\pi}{x}) : x \in \mathbb{R},\ x \neq 0\}$.
Find S'.

<u>Closed set</u>: A set $S \subset \mathbb{C}$ is said to be closed if S contains each of its limit points. That is, $S' \subset S$.

Ex. Show that a set $S \subset \mathbb{C}$ is closed iff S^c is open. (18)

Suppose S is closed, but S^c is not open.

Then $\exists\, z \in S^c$ s.t. $\forall\, r > 0$,

$$B(z, r) \not\subset S^c$$

$$\Rightarrow B(z, r) \cap S \neq \Phi$$

$$\Rightarrow \left(B(z, r) \setminus \{z\}\right) \cap S \neq \Phi, \ \forall\, r > 0$$

$$\Rightarrow z \in S' \subset S \ (\because S \text{ is closed})$$

is a contradiction.

On the other hand, suppose S^c is open, but S is not closed.

That is, $S' \not\subset S$

$$\Rightarrow \exists\, z \in S' \text{ but } z \notin S$$

Since $z \notin S \Rightarrow z \in S^c$ (open)

$$\Rightarrow \exists\, r > 0 \text{ s.t. } B_r(z) \subset S^c.$$

$\Rightarrow B(z,r) \cap S = \Phi$

$\Rightarrow (B(z,r) \setminus \{z\}) \cap S = \Phi$ (19)

$\Rightarrow z$ is not a limit point. X

Closure of a set:

The closure of a set is defined by

$\bar{S} = S \cup S'$

Ex. Show that a set $S \subset \mathbb{C}$ is closed iff $\bar{S} = S$.

Ex. Show that $\bar{S} = S^{\circ} \cup \partial S$.

Note that $S^{\circ} \cap \partial S = \Phi$. If $z \in S^{\circ} \cap \partial S$, then for $z \in S^{\circ}$ $\exists\, B_r(z) \subset S$. But $z \in \partial S$

$\Rightarrow B_r(z) \cap S^c \neq \Phi$

which is absurd. Hence

$S^{\circ} \cap \partial S = \Phi$.

Claim $S^\circ \cup \partial S = S \cup S'$

Note that $S^\circ \subset S$. (20)

If $z \in \partial S$ and $z \notin S$

$\Rightarrow (B(z,r) \setminus \{z\}) \cap S \neq \Phi$, $\forall r > 0$

$\Rightarrow z \in S'$

Hence $S^\circ \cup \partial S \subset S \cup S' = \overline{S}$.

To prove $S \cup S' \subset S^\circ \cup \partial S$, let $z \in S \cup S'$. If $z \notin S^\circ$ and $z \notin \partial S$

then $\exists\, r > 0$ s.t. $B(z,r) \cap S = \Phi$

$\Rightarrow (B(z,r) \setminus \{z\}) \cap S = \Phi$

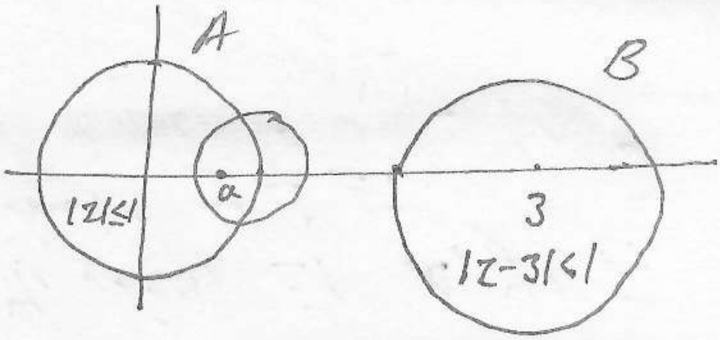
$\Rightarrow z \notin S'$

which is a contradiction.

Hence $S^\circ \cup \partial S = \overline{S}$.

Connected Sets:

Let us consider a space



$X = \{z \in \mathbb{C} : |z| \le 1\}$

$\quad \cup \{z \in \mathbb{C} : |z-3| < 1\} = A \dot{\cup} B$ (21)

with $d(z,w) = |z-w|$, when $z, w \in X$.

Then (X, d) is a metric space, and the set $A = X \setminus B$ is ~~op~~ closed as B is open in X. On the other hand for $a \in A$,

$B(a,1) \cap (A \dot{\cup} B) = B(a,1) \cap A \subset A$

$\Rightarrow$ A is an open set too.

Similarly, $B = \{z : |z-3| < 1\}$ is open & closed both.

Defn: A subset $S \subset \mathbb{C}$ is said to be disconnected if $\exists$ two open disjoint sets U & V of $\mathbb{C}$ and subset $A, B \subset S$ such that $S = (A \cap U) \dot{\cup} (B \cap V)$.

Clearly $S \subset U \dot{\cup} V$. That is

S is partitioned by two open sets U & V in $\mathbb{C}$. (22)

Clearly, the above space (X, d) is disconnected. Note that singleton set is always connected.

* A set $S \subset \mathbb{C}$ is said to be connected if it is not disconnected.

It is also clear that $S \subset \mathbb{C}$ is connected if the only clopen (closed & open) sets in S is either ϕ or S (in the relative topology of S).

<u>Lemma:</u> A set $E \subset \mathbb{R}$ is connected iff E is an interval.

Proof: Suppose E is ~~no~~ connected and not an interval, then $\exists$ $x, y \in E$ st $x < z < y$, but $z \notin E$.

Then $E \subset (-\infty, z) \cup (z, \infty)$.

Hence E is disconnected.

Hence, E is disconnected.

Conversely, let $E = [a, b]$, $a, b \in \mathbb{R}$

Suppose $A \subsetneq E$ and $a \in A$, A open.

We claim A is not closed.

Since A is open, and $a \in A$, $\exists\, \epsilon > 0$ s.t. $[a, a+\epsilon) \subset A$.

Let $\gamma = \sup\{\epsilon : [a, a+\epsilon) \subset A\}$.

Claim $[a, a+\gamma) \subset A$.

Let $a \leq x < a+\gamma$. Let $h = a+\gamma-x > 0$.

Then by def^n of supremum, $\exists\, \epsilon > 0$ s.t. $\gamma - h < \epsilon < \gamma$ and

$[a, a+\epsilon) \subset A$.

But $a \leq x = a+\gamma-h < a+\epsilon \Rightarrow x \in A$.

However, $a+\gamma \notin A$. If on contrary, $a+\gamma \in A$, then $[a+\gamma, a+\gamma+\delta) \subset A$, because A is open. But this contradicts the def^n of γ.

$\Rightarrow$ A is not a closed set.
For other connected intervals, the similar proof is go through.

For $z, w \in \mathbb{C}$, the line joining z & w is the set

$$[z,w] := \{tw + (1-t)z : 0 \le t \le 1\}.$$

A polygon in the complex plane is union of continued line segments. Let it a polygon joining a & b is $P = \bigcup_{k=1}^{n} [z_k, w_k]$, where
$a = z_1$, $b = w_n$, $w_k = z_{k+1}$,
$1 \le k \le n-1$.
We also denote P by $P = [a, z_1 \cdots z_n\, b]$.

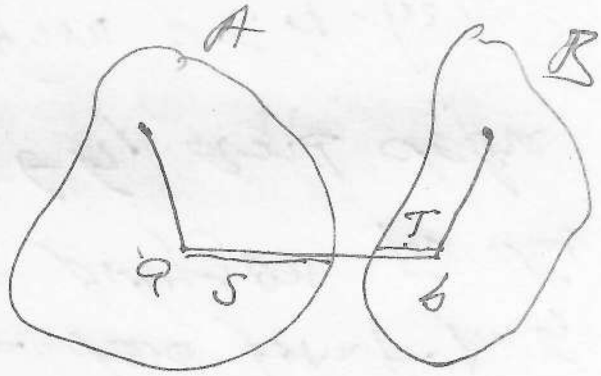
Defn: A set $S \subset \mathbb{C}$ is said to be path connected if any two points in S can be joined by a polygon.

<u>Theorem</u>: A open set G is conn. iff G is path conn.

Proof: Suppose G is path conn. But not connected. Then $\exists$ open sets A & B s.t. $G = A \cup B$, $A \cap B = \Phi$, $A \neq \phi \neq B$.

By hypothesis, $\exists$ a polygon P from $a \in A$ to $b \in B$. It is clear that P has one point in A and another point in B. So we can assume that

that $P = [a, b]$.



define

$S = \{ s \in [0,1] :$
$sb + (1-s)a \in A \}$

& $T = \{ t \in [0,1] : tb + (1-t)a \in B \}$.

Then $S \cup T = [0,1]$, $0 \in S$, $1 \in T$.

Hence we get a disconnection of $[0,1]$

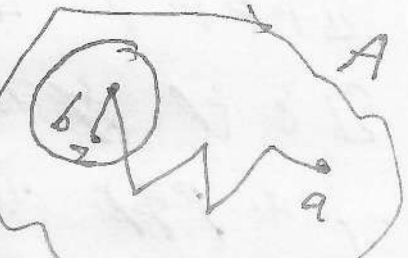
because S & T are open in $[0,1]$, which is a contradiction. Hence G is connected.

Conversely, suppose that G is connected. We show that every two point in G can be joined by a polygon. For this, ~~let~~ fix a point $a \in G$, and define

$$A = \{ b \in G : \exists \text{ polygon } P \subset G \text{ from } a \text{ to } b \}$$

We claim A is both open & closed.

To show A is open, let $b \in A$ and $P = [a, z_1, \ldots, z_n, b]$ be a polygon from a to b.



Since $b \in G$ and G is open, $\exists\, \epsilon > 0$ s.t. $B(b, \epsilon) \subset G$.

~~s.t.~~ If $z \in B(b, \epsilon)$, then

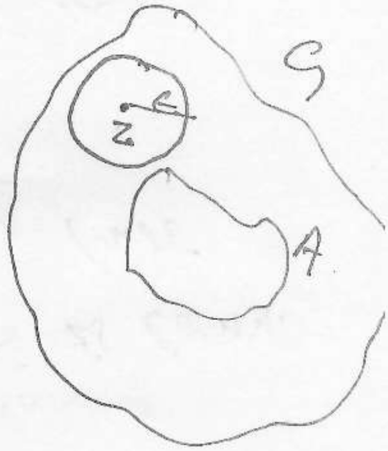
Then $[b, z] \subset B(b, \epsilon) \subset G$.

Hence polygon $Q = P \cup [b, z] \subset G$,
which is $Q = P[a, z]$

$$\Rightarrow B(b, \epsilon) \subset A \Rightarrow A \text{ open.}$$

To show A is closed, we show $G \setminus A$ open.

Let $z \in G \setminus A$. Then $\exists\, \epsilon > 0$
s.t. $B(z, \epsilon) \subset G \setminus A$



If $\exists\, b \in A \cap B(z, \epsilon)$, then
as above we can construct
a polygon from a to z, but
$z \notin A \Rightarrow \quad B(z, \epsilon) \cap A = \Phi$

$$\Rightarrow B(z, \epsilon) \subset G - A.$$

$$\Rightarrow G \setminus A \text{ is open} \Rightarrow A \text{ is closed.}$$

Thus $A = G$.

<u>Thm:</u> A set $S \subset \mathbb{C}$ is disconnected iff
$\exists\, f : S \xrightarrow[\text{onto}]{\text{Cont}} \{0, 1\}$ (two point discrete
space).

pf: $f: S \longrightarrow \{0,1\}$ cont & onto, then (28)

$A = f^{-1}\{0\}$ & $B = f^{-1}\{1\}$ are non-empty disjoint open sets & $A \cup B = S$.

Conversely, if $S = A \cup B$, where A & B are non-empty disjoint open sets in S, then by setting $f(A) = \{0\}$ & $f(B) = \{1\}$, we can define a continuous onto map

$$f: S \longrightarrow \{0,1\},$$

This gives a perfect replacement of defn of conn. set.

Thus, we conclude that S is conn. iff every cont map from S into a discrete space is constant.

Thm: Let $f: S \subset \mathbb{C} \longrightarrow \mathbb{C}$ be cont. If S is conn., then $f(S)$ is connected.

pf: suppose $f(S)$ is not connected. Then $\exists\ g: f(S) \xrightarrow[\text{onto}]{\text{cont}} \{0,1\}$.

onto

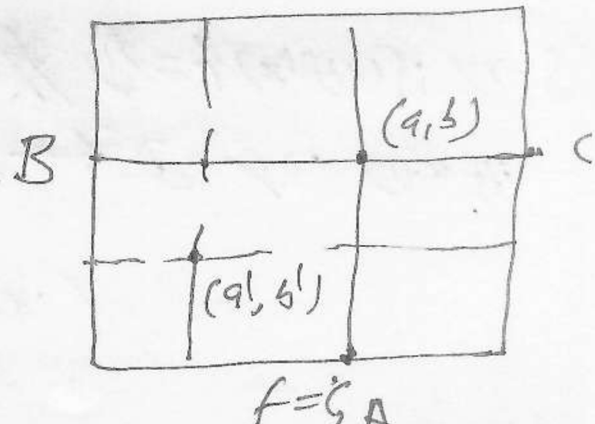
$\Rightarrow g \circ f : S \xrightarrow[\text{onto}]{\text{Cont}} \{0,1\}$

$\Rightarrow S$ is disconn.

Prop: Let A, B be two conn. subsets of $\mathbb{C}$, then $A \times B$ is conn. in $\mathbb{C}^2$.

Pf: Suppose $\exists\ f : A \times B \longrightarrow \{0,1\}$ is conti.
we claim f is constant.
For $(a,b) \in A \times B$, $f(a,\cdot)$ and $f(\cdot,b)$ are cont. f^n on A & B respy.
Since A & B are conn, it follows that $f(a,\cdot)$ and $f(\cdot,b)$ are constants.

That is f is constant on every vertical & horizontal lines.



Hence, f is constant on $A \times B$. Thus, $A \times B$ is conn.

(Hint: $f(a,b) = c_1 = c_2$
$f(a',b') = c_1 = c_2$)

Ex. Show that $(0,1)\times(0,1)$ cannot be written as disjoint union of countably many open balls. (30)

(Hint: $(0,1)\times(0,1)$ is conn.)

Ex. Let $S\subseteq\mathbb{C}$ & $f:S\to\mathbb{C}$ be conti. Show that S is conn. iff $G_f=\{(z,f(z)) : z\in S\}$ is conn. in $\mathbb{C}^2$.

Define $g:S\to S\times S$, by $g(z)=(z,f(z))$.

Then g is conti & $G_f=g(S)$ is conn, since S is conn.

On the other hand, projection

$$p_1 : G_f \to S$$

$$p_1(z,f(z)) = z \quad \text{is cont}$$

$\Rightarrow$ S is conn. as $p_1(G_f)=S$.

Define $g : (0,1] \to [-1,1]$ by $g(x) = \sin \pi/x$.
Then g is Cont. $\Rightarrow$ $g((0,1])$ is Conn.

(32)

Note that $G_g \subseteq G_f \subseteq \overline{G_g}$.
$\Rightarrow G_f$ is Conn.

Def^n: A maximal Conn. subset of a set $S \subset \mathbb{C}$ is called Component.
(i.e. If $D \subset S$ is a Component, then $\nexists$ any proper subset of D which is Conn.)

Ex. $X = \{|z| \leq 1\} \cup \{|z-3| < 1\}$
has two Components $\{|z| \leq 1\}$ and $\{|z-3| < 1\}$.

* It is easy to see that any two distinct components of a space S is always disjoint.

Ex. If $A \subset \mathbb{C}$ is conn., then for $A \subseteq B \subseteq \bar{A}$, it implies that B is conn. In particular, $\bar{A}$ is conn.

(i.e. a ~~set~~ set containing some or all limit points of a conn. set is conn.)

Suppose $f: B \xrightarrow[\text{onto}]{\text{cont}} \{0,1\}$. Then $f: A \xrightarrow{\text{cont}} \{0,1\}$ since A is conn., f is const on A

$\Rightarrow \tilde{f}: \bar{A} \to \mathbb{C}$ with $\tilde{f}/A = f$ is constant

$\Rightarrow \tilde{f}$ is constant on B.

(note every unif cont function on S can be extended unif'ly to $\bar{A}$).

Ex. let $f: [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin \pi/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

[Topologist's sine curve]

Show that f is not ~~connected~~ continuous but G_f is connected.

Sequence, Limit and Continuity

For $S \subseteq \mathbb{C}$, a complex-valued function is a rule that assign each $z \in S$ a unique complex number in $\mathbb{C}$.

i.e. $f : S \subseteq \mathbb{C} \longrightarrow \mathbb{C}$.

Then $f(z) \in \mathbb{C}$

$\Rightarrow f(z) = u(z) + i v(z)$,

where $u(z), v(z) \in \mathbb{R}$. That is,

$u, v : S \longrightarrow \mathbb{R}$.

Complex Sequence:

A seq^n is a map $f : \mathbb{N} \longrightarrow \mathbb{C}$, which we write as $f(1), f(2), \ldots f(n), \ldots$ we denote $z_n = f(n)$.

Defn: A seq^n z_n is said to converge to $z \in \mathbb{C}$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|z_n - z| < \epsilon \;\; \forall n \geq N$.

In this case we write

$$\lim_{n\to\infty} z_n = z.$$

If $z_n = x_n + i y_n$ & $z = x + iy$.
Then it is easy to see that

$$\lim z_n = z \text{ iff } \lim x_n = x \text{ \& } \lim y_n = y.$$

Limit of a function:

Suppose $f: B_r(z_0) \setminus \{z_0\} \longrightarrow \mathbb{C}$.
we say that f has limit l at z_0
if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ st

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$$

In this case we write

$$\lim_{z\to z_0} f(z) = l.$$

Note that if the limit of a function exists at point, then it is unique.

If we write

$$f(z) = u(x,y) + i v(x,y),$$

$z = x+iy$, $z_0 = x_0 + i y_0$ and

$\lim_{z \to z_0} f(z) = \alpha + i\beta$ iff

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \alpha \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = \beta.$$

Note that the point z_0 can be approached through any path. If the limit $\lim_{z\to z_0} f(z)$ exists, it must be independent of path approaching z_0.

Result: Let $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be such that $\lim_{z\to z_0} f(z) = \alpha \neq 0$. Then $\lim_{z\to z_0} \frac{1}{f(z)}$ exists & $= \frac{1}{\alpha}$.

Proof: Since $\lim_{z\to z_0} f(z) = \alpha \neq 0$, $\exists$ an

Small deleted nbd $B_r(z_0) \setminus \{z_0\}$ s.t. (36)

$$f(z) \neq 0 \quad \forall \; z \in B_r(z_0) \setminus \{z_0\}.$$

Also, for $\epsilon > 0$, $\exists \; \delta > 0$ s.t.

$$|f(z) - \alpha| < \epsilon \quad \text{for } 0 < |z - z_0| < \delta < r.$$

Now, $\left| \frac{1}{f(z)} - \frac{1}{\alpha} \right| = \left| \frac{f(z) - \alpha}{f(z)\,\alpha} \right| < \frac{\epsilon}{|f(z)|\,|\alpha|}$

$$< \frac{\epsilon}{(|\alpha| - \epsilon)\,|\alpha|} < \frac{\epsilon}{(|\alpha| - \epsilon)^2} \to 0 \quad \text{as } \epsilon \to 0.$$

Thus, $\lim \frac{1}{f(z)} = \frac{1}{\alpha}$.

<u>Continuity of a function</u>

Let $S \subset \mathbb{C}$ and $f : S \longrightarrow \mathbb{C}$.

We say that f is cont. at $z_0 \in S$ if $\forall \epsilon > 0$, $\exists \; \delta > 0$ s.t.

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } z \in S \text{ with } |z - z_0| < \delta.$$

That is, f is ~~cont~~ continuous at z_0 if $\lim_{z \to z_0} f(z)$ exists & $= f(z_0)$.

$z \to z_0$ ... $\to f(z_0)$

Proposition: Let $f : S \to \mathbb{C}$. Then f is cont. at $z_0 \in S$, iff $\forall$ seq^n $z_n \to z$ $\Rightarrow f(z_n) \to f(z_0)$.

Proof: Suppose f is cont. at z_0. Then for $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon. \quad \text{—} (*)$$

Let $z_n \to z_0$. Then $\exists\ N \in \mathbb{N}$ st

$$n \geq N \Rightarrow |z_n - z_0| < \delta.$$

But from $(*)$, it follows that

$$|f(z_n) - f(z_0)| < \epsilon, \text{ whenever } n \geq N.$$

Hence, $f(z_n) \to f(z_0)$.

Conversely, Suppose f is not cont at z_0. Then $\exists\ \epsilon_0 > 0$ st $\forall\ \delta > 0$, $\exists\ z \in S$ with $|z - z_0| < \delta$ but $|f(z) - f(z_0)| \geq \epsilon_0$. Set $\delta = 1/n$. Then $\exists$ seq^n z_n st

$|z_n - z_0| < 1/n$ but $|f(z_n) - f(z_0)| \geq \epsilon_0$ which is a contradiction. (38)

* If $f : S \subset \mathbb{C} \longrightarrow \mathbb{C}$ be cont. at z_0 & $f(z_0) \neq 0$, then $1/f$ is cont. at z_0.
(Hint: the proof follows similar to limit of $1/f$ etc).

* If $f = u + iv$. Then f is cont at z_0 iff u & v cont at z_0.

* If $f : S \rightarrow \mathbb{C}$ is cont at z_0 & $f(z_0) \neq 0$. Then $\exists \, \delta > 0$ s.t.
$$f(z) \neq 0 \;\; \forall \, z \in B_\delta(z_0) \cap S.$$
(Hint: Take $\epsilon = \frac{1}{2} |f(z_0)| > 0$).

Defn: A function $f : S \longrightarrow \mathbb{C}$ is said to be uniformly continuous if $\forall \, \epsilon > 0$, $\exists \, \delta > 0$ st

$|f(z)-f(z_0)| < \epsilon$ whenever $|z-z_0| < \delta$

Thm: If f is uniformly cont on S. Then $\exists\ \tilde{f} : \overline{S} \to \mathbb{C}$ which is unif cont on $\overline{S}$ where $\tilde{f}|_S = f$.

Defn: A set S in $\mathbb{C}$ is said to be compact if S is closed & bounded in $\mathbb{C}$.

Result: Show that continuous image of compact set is compact.

proof: let $f : K\,(\text{cpt}) \subseteq \mathbb{C} \xrightarrow{\text{cont}} \mathbb{C}$.

Then claim $f(K)$ is closed. let $f(z_n) \to w$. Since $z_n \in K$ & K is compact, $\exists\ z_{n_k} \to z \in K$ (by B-W T),

$\Rightarrow w = \lim f(z_{n_k}) = f(z)$.

$\Rightarrow f(K)$ is closed.

Suppose $f(K)$ is not bounded.

Then for each $n \in \mathbb{N}$, $\exists\ z_n \in K$ st

$$|f(z_n)| > n$$

But again, $z_n \in K$ & $\exists\ z_{n_k} \to z \in K$.

$$\Rightarrow |f(z_{n_k})| > n_k$$

$$\Rightarrow |f(z)| = \infty \quad \times$$

(41)