Complex differentiation

Before we start, we note that complex differentiation has far reaching consequences than real derivatives. In fact complex differentiation is a very particular case of real differention, in which partial derivatives satisfy Cauchy-Riemann equations.

Let D be an open set in $\mathbb{C}$.

A function $f : D \longrightarrow \mathbb{C}$ is said to be complex differentiable at $z_0 \in D$ (or simply differentiable) if

$$\lim \frac{f(z_0+h)-f(z_0)}{h} \text{ exists.}$$

The limit is denoted by $f'(z_0)$.

Ex. $f(z) = z^2$. Then

$$f'(z) = \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h} = 2z.$$

Similarly, if $f(z) = z^n$, then

$$f'(z) = n z^{n-1}$$

Ex. $f(z) = \bar{z}$ is not differentiable at any point of $\mathbb{C}$.

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\bar{h}}{h}$$

does not exist.

However, if we realize f as f^n on $\mathbb{R}^2$, then $f(x,y) = (x,-y)$ is everywhere differentiable and

$$f'(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x,-y).$$

But we write $u = x$, $v = -y$

then $u_x = 1 \neq -1 = v_y$, etc.

$u_x = 1 \neq -1 = v_y$, ...

Algebra of differentiation:

If f & g are diff then

(i) $\alpha f + \beta g$ is differentiable

(ii) fg is diff. (iii) f/g is diff. if $g \neq 0$.

Ex. $f : \mathbb{C} \to \mathbb{R}$, $f(z) = |z|^2 = x^2+y^2$

Then
$$\lim_{h \to 0} \frac{|z_0+h|^2 - |z_0|^2}{h} = \lim_{h\to 0} \frac{z_0\bar{h} + \bar{z}_0 h + h\bar{h}}{h}$$
$$= z_0 \lim_{h \to 0} \frac{\bar{h}}{h} + \bar{z}_0 + \bar{h}$$

This limit exists iff $z_0 = 0$.

However, if we consider
$$f : \mathbb{R}^2 \to \mathbb{R} \text{ by}$$
$$f(x,y) = x^2+y^2.$$
Then f is differentiable everywhere.

Also, $f(x,y) = (x^2+y^2, 0) = (u,v)$

$$f' = \begin{pmatrix} 2x & 2y \\ 0 & 0 \end{pmatrix}.$$ Here $u_x = 2x \neq 0 = v_y$

Now, observe that

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

is differentiable if

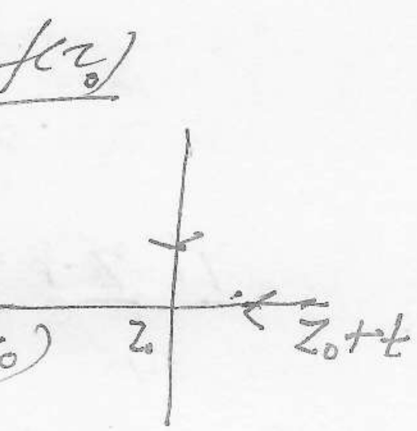
$$f(z+h) - f(z) = h\, f'(z) + h\, \epsilon(h) \quad \text{—}(*)$$

i.e. $f'(z)$ is a $\mathbb{C}$-linear map from $\mathbb{C}$ to $\mathbb{C}$.

Now, let $f = u + iv$, then

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

By taking $h = t$, $t \to 0$, we get $f'(z_0) = u_x(z_0) + i\, v_x(z_0)$



while $h = it$, $t \to 0$, we get

$$f'(z_0) = v_y(z_0) + i(-u_y(z_0)).$$

But then, $u_x(z_0) = v_y(z_0)$ & $v_x(z_0) = -u_y(z_0)$.

Now, consider $f : \mathbb{R} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

by $f(x,y) = (u(x,y), v(x,y))$.

by $f(x,y) = (u(x,y), v(x,y))$.

Then $f'(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$ (46)

$$\cong u_x + i v_x$$

Thus, every $\mathbb{C}$-linear map is an $\mathbb{R}$-linear map but converse need not be true.

Also, $\mathbb{R}$-linear is $\mathbb{C}$-linear iff Cauchy-Riemann equations are satisfied.

Polar form of C-R equations:

Consider $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$ (By chain rule)

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$= \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$

$$= \frac{1}{r}\left\{\left(\frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)\right)\right\}$$

$$= \frac{1}{r}\left\{\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}\right\} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Ex. Show that $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

satisfies the C-RE's but not differentiable at $z=0$.

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{(x,y)\to(0,0)} \frac{\left(\frac{x^3-3xy^2}{x^2+y^2} + i\,\frac{y^3-3x^2y}{x^2+y^2}\right)}{x+iy}$$

for $z = x \to 0$, we get limit

$$\lim_{(x,0)\to(0,0)} \frac{x-0}{x-0} = 1$$

let z approaches to 0 along $y=x$, then

$$\lim_{(x,x)\to(0,0)} \frac{-x-iy}{x+iy} = -1.$$

Hence, f is not differentiable at $z=0$.

Now, $u_x(0,0) = \lim_{x\to 0} \frac{x-0}{x} = 1$, $u_y(0,0) = 0$

$v_x(0,0) = 0$ & $v_y(0,0) = 1$.

ie $u_x = v_y$ & $u_y = -v_x$ satisfied.

i.e. $u_x = v_y$ & $u_y = -v_x$ ...

Chain rule for differentiation:

Let D_1, D_2 be two open sets in $\mathbb{C}$ s.t. $f : D_1 \to \mathbb{C}$, $f(D_1) \subset D_2$ and $g : D_2 \to \mathbb{C}$. If f & g both are differentiable then $g \circ f : D_1 \to \mathbb{C}$ is differentiable and

$$(g \circ f)'(z) = g'(f(z)) f'(z).$$

Proof:

Let $$\eta(h) = \frac{(g \circ f)(z+h) - g \circ f(z) - g'(f(z)) f'(z) h}{h}$$

write $w = f(z)$ & $K = f(z+h) - f(z) \to 0$ as $h \to$

$$\eta(h) = \frac{g(w+K) - g(w) - g'(w) f'(z) h}{h}$$

$$= \frac{g(w+K) - g(w) - g'(w)K + g'(w)K - g'(w)f'(z)h}{h}$$

$$= \frac{K\eta_g(K) + g'(w)(K - f'(z)h)}{h}$$

$$\eta(h) = \frac{k\,\eta_g(k) + g'(w)\,\eta_f(h)\,h}{h}$$

$$= \frac{k}{h}\,\eta_g(k) + g'(w)\,\eta_f(h)$$

Note that $k = f(z+h) - f(z) = h f'(z) + h\,\eta_f(h)$

$\therefore \eta(h) = \left(f'(z) + \eta_f(h)\right)\eta_g(k) + g'(w)\,\eta_f(h)$

$\longrightarrow 0$ as $h \to 0$ $\quad (\because h \to 0 \Rightarrow k \to 0)$

Cor: If $f : \Omega_1 \longrightarrow f(\Omega_1) \subset \Omega_2 \xrightarrow{g} \mathbb{R}$.
Then $g \circ f : \Omega_1 \longrightarrow \mathbb{R}$. If f & g are
diff. then $(g \circ f)'(z) = g'(f(z))\,f'(z)$.

Sufficient condition for differentiability of a function:

Suppose f be defined on an open set D in $\mathbb{C}$ & $f = u + iv$. If u, v satisfying C-R equations and having 1st order partial derivative cont., then

f is differentiable.

proof: without loss of generality we can assume that D is a ball, else for $z \in D$, $\exists\, r > 0$ st $B(z, r) \subset D$, as long as differentiation at z is concerned. Now, let $D = B(0, r)$.

Consider
$$\frac{f(h+ik) - f(0)}{h+ik}$$
$$= \frac{u(h,k) - u(0,0) + i\,(v(h,k) - v(0,0))}{h+ik}$$
$$= \frac{h\,u_x(\theta_1 h, \theta_1 k) + k\,u_y(\theta_1 h, \theta_1 k) + i\,(h\,v_x(\theta_2 h, \theta_2 k) + k\,v_y(\theta_2 h, \theta_2 k))}{h+ik} \quad \text{(by MVT)}$$

Now,
$$\frac{f(h+ik) - f(0)}{h+ik} - (u_x(0,0) + i\,v_x(0,0))$$
$$= \frac{\alpha + i\beta}{h+ik} - (u_x + i\,v_x) \to 0 \text{ as } h+ik \to 0$$

Analytic function:

A function f is said to be analytic at point $z_0 \in \mathbb{C}$ if $\exists\ r > 0$ st f is diff. on $B_r(z_0)$.

And f is called analytic on an open set $G \subset \mathbb{C}$ if f is analytic at each point of G. Note that it is equivalent that f is diff. on G.

Analyticity do requires on a nhd due to two prominent reasons: power series rep^n of analytic function and Cauchy integral formula.

* A function f is said to be entire if it is analytic on whole complex plane.

* The function $f(z) = \frac{1}{z}$ if $z \neq 0$ is analytic on $\mathbb{C} \setminus \{0\}$ but not

on $\mathbb{C}$. Hence $f(z)=1/z$, $z\neq 0$ is not an entire function. (52)

Ex. The function $f(z)=|z|^2$ is diff. at $z=0$ but not analytic ~~at z=0~~ at any point.

* If f is analytic on an open set D and $f'\neq 0$, then $1/f$ is ~~dif~~ analytic.
* Similarly Composition of two analytic function is analytic.

Ex. 2^{3^z} is analytic, when 2^z & 3^z have considered for their principle values.

$f(z)=2^z = e^{z\log 2}$ is analytic

& $g(z)= 3^z = e^{z\log 3}$ is analytic.

Hence $(g\circ f)(z)= g(2^z)=2^{3^z}$.

$(g\circ f)'(z)= g'(f(z))f'(z)$.

* An open & connected set in $\mathbb{C}$ is called domain. (53)

* If f is analytic on domain D, be such that either real part, or imaginary part or argument is constant, then f is constant.

Suppose $f = u + iv$, $u = $ const

$$0 = u_x = v_y \ \& \ 0 = u_y = -v_x$$

Note that since f is diff. on D, it follows that u & v are diff. on D.

Since $v : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable, by MVT

$$v(x, y) - v(x', y') = (x - x') v_x(c) + (y - y') v_y(c) = 0$$

$\Rightarrow$ v is const

$\Rightarrow$ f is constant on D

Similarly, if v is const.

Now, if $f(z) = |f(z)| e^{i\alpha}$ α fixed.

(i.e. argument of f is constt),

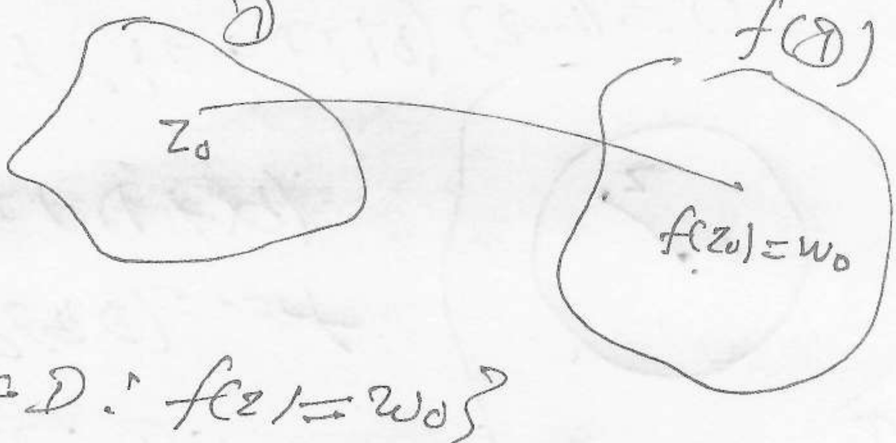
then $e^{-i\alpha} f(z) = |f(z)| + i\,0$ (54)

By previous part, $e^{-i\alpha} f(z) = $ constt.

<u>Result</u>: If $f : D$ (domain) $\to \mathbb{C}$
is analytic & $f'(z) = 0$ on D,
then f is constant.

<u>proof</u>:

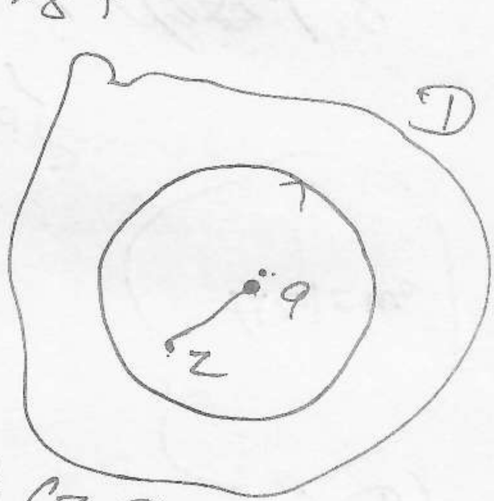
Fix $z_0 \in D$,
and write
$w_0 = f(z_0)$.



Put $A = \{ z \in D : f(z) = w_0 \}$

Then we claim that A is closed & open both. Since D is connected, in this situation $A = D$.

A is closed because f is continuous.
Next, we show that A is open. (55)
Fix $a \in A$, then $\exists\, \epsilon > 0$ s.t.
$B(a, \epsilon) \subset D$.



Let $z \in B(a, \epsilon)$, set
$g(t) = f(tz + (1-t)a)$.
Then $g'(t) = f'(tz + (1-t)a)\,(z-a) = 0$.
$\Rightarrow g'(t) = (g_1'(t)\, g_2'(t)) = 0$
$\Rightarrow g_1' = 0 = g_2'$
$\Rightarrow g$ is constant on $[0,1]$.
Now, $f(z) = g(1) = g(0) = f(a) = w_0$
$\Rightarrow z \in A \Rightarrow B(a, \epsilon) \subset A$
Hence A is open.

Harmonic function:

A function φ defined on an open set D in $\mathbb{C}$ is said to be harmonic if

$$\varphi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

has 2nd order partial derivatives continuous & satisfies

$$\varphi_{xx} + \varphi_{yy} = 0 \text{ on } D$$

i.e. $\Delta \varphi = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Harmonic Conjugate:

A function $v : D \longrightarrow \mathbb{R}$ is said to be harmonic conjugate of $u : D \longrightarrow \mathbb{R}$ (harmonic) if $f = u + iv$ is analytic.

Ex. If $u(x,y) = x^2 - y^2$, then $\Delta u = 0$

$$v_y(x,y) = u_x(x,y) = 2x$$

$$\Rightarrow \quad v(x,y) = \int_0^y 2x \, dy + \varphi(x)$$

$\Rightarrow v(x,y) = 2xy + \varphi(x)$.

Also, $v_x(x,y) = -u_y(x,y) = +2y$

$$\Rightarrow 2y + \varphi'(x) = 2y \Rightarrow \varphi'(x) = 0$$

$$\Rightarrow v(x,y) = 2xy + c$$

* Harmonic conjugate if exists is unique upto a constant.

If v_1 & v_2 are two harmonic conjugate then

$$f_1 = u + iv_1 \ \& \ f_2 = u + iv_2$$

both are analytic

$$\Rightarrow f_1 - f_2 = i(v_1 - v_2) \text{ is analytic}$$

$$\Rightarrow f_1 - f_2 = \text{const} = ic$$

$$\Rightarrow v_1 = v_2 + c.$$

It is not necessary that given a harmonic function has always harmonic conjugate.

Conjugate.

Ex. $u(x,y) = \log(x^2+y^2)^{1/2}$ if $x^2+y^2 \neq 0$ (58)

is harmonic on $\mathbb{C}\setminus\{0\}$, but has no harmonic conjugate.

The polar form of Laplacian is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$u(r,\theta) = \log r \Rightarrow \Delta u = 0.$$

Suppose $f(r,\theta) = \log r + i v(r,\theta)$,

and f is analytic. Then v satisfies CR Equations.

$$\frac{1}{r} = \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \Rightarrow \frac{\partial v}{\partial \theta} = 1$$

$$\Rightarrow v(r,\theta) = \theta + \varphi(r)$$

But $\boxed{} = \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \Rightarrow \varphi'(r) = 0$

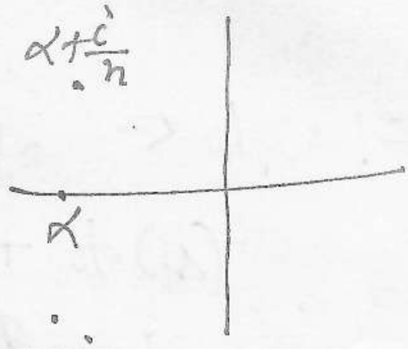
$$\Rightarrow v = \theta + c.$$

w.l.g. take $c = 0$, then

$$f(r,\theta) = \log r + i\theta,$$

i.e $f(z) = \log|z| + \mathrm{Arg}(z)$

But $\mathrm{Arg}(z)$ is not continuous on $(-\infty, 0)$.



$\alpha + \frac{i}{n} \longrightarrow \alpha$

$\alpha - \frac{i}{n} \longrightarrow \alpha$

But $\mathrm{Arg}(\alpha + \frac{i}{n}) = \tan^{-1}\frac{\alpha}{n}$ $\alpha + \frac{i}{n}$

$\longrightarrow \cancel{2\pi} - 0 = \pi$

But $\mathrm{Arg}(\alpha - \frac{i}{n}) = -\pi - \tan^{-1}\frac{\alpha}{n} \longrightarrow -\pi$.

($\because \alpha = -\beta$, $\beta > 0$).

<u>Theorem:</u> If the domain is either $B_r(0)$ or $\mathbb{C}$, then every harmonic function u has harmonic conjugate.

Pf: Let $u : B_r(0) \longrightarrow \mathbb{R}$ be harmonic. If v is harmonic conjugate, then u, v satisfy

CR-E.

$$v_y = u_x \Rightarrow v(x,y) = \int_{s=0}^{y} u_x(x,s)\,ds + \varphi(x)$$

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$$\Rightarrow v_x(x,y) = \int_{s=0}^{y} u_{xx}(x,s)\,ds + \varphi'(x)$$

$$= -\int_{s=0}^{y} u_{yy}(x,s)\,ds + \varphi'(x)$$

$$= -\left(u_y(x,y) - u_y(x,0)\right) + \varphi'(x)$$

$$\Rightarrow \varphi'(x) = -u_y(x,0)$$

$$\Rightarrow v(x,y) = \int_{s=0}^{y} u_x(x,s)\,ds - \int_{t=0}^{x} u_y(t,0)\,dt$$

note that in the above proof, we use integration parallel to either axis, which possible due to specialty in the domain.

<u>Theorem</u>: Let G, Ω be two open sets in $\mathbb{C}$ s.t. $f: G \to f(G) \subset \Omega \xrightarrow{g} \mathbb{C}$.
If f & g both are cont. $g \circ f(z) = z$ and g is diff. & $g'(z) \neq 0$, then

f is differentiable and $g'(f(a))f'(a) = 1$ (61)

Proof: $$\frac{g(f(a+h)) - g(f(a))}{h} = 1$$

$$\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \frac{f(a+h) - f(a)}{h} = 1$$

Since f is conti; $\lim_{h \to 0} [f(a+h) - f(a)] = 0$

$$\Rightarrow \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} = g'(f(a))$$

Since $g'(f(a)) \neq 0 \Rightarrow \lim \frac{f(a+h) - f(a)}{h}$ has to exist & hence

$$g'(f(a))f'(a) = 1.$$