Power Series

For $a_n \in \mathbb{C}$, the series $\sum a_n$ is said to converge z if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ st
$$\left|\sum_{n=0}^{m} a_n - z\right| < \epsilon, \ \forall\, m \geq N.$$

St $\left|\sum_{n=m+1}^{\infty} a_n\right| < \epsilon$ for $m \geq N$ $\left(\because z = \sum_{n=0}^{\infty} a_n\right)$.

* $\sum a_n$ is said to be absolutely conv. if the series $\sum |a_n|$ is conv.

* Every abs. conv series is conv.

Let $\epsilon > 0$, & $z_n = a_1 + \cdots + a_n$

Since $\sum |a_n|$ is conv, for $\epsilon > 0$ $\exists N \in \mathbb{N}$ st $\sum_{n=m+1} |a_n| < \epsilon$, $m \geq N$.

If $m \geq k \geq N$, then
$$|z_m - z_k| = \left|\sum_{n=k+1}^{m} a_n\right| \leq \sum_{n=k+1}^{m} |a_n| < \sum_{n=k+1}^{\infty} |a_n| <$$

Hence z_n is a Cauchy seq^n, Hence $\exists\, z \in \mathbb{C}$ s.t. $z_n \to z$. 63

Hence $\sum a_n = z$.

For $a_n \in \mathbb{R}$, we write

$$\liminf a_n = \sup_{n \geq 1}\left(\inf_{k \geq n} a_k\right) = \underline{\lim}\, a_n$$

& $$\limsup a_n = \inf_{n \geq 1}\left(\sup_{k \geq n} a_k\right) = \overline{\lim}\, a_n.$$

Note that

(i) $\liminf a_n \leq \limsup a_n$

(ii) $\limsup (a_n b_n) \leq (\limsup a_n)(\limsup b_n)$

(iii) $\liminf (a_n b_n) \geq \liminf (a_n) \liminf (b_n)$.

(Hint: $a_n b_n \geq a_n \inf_{k \geq n} b_k \geq \inf_{k \geq n} a_k \inf_{k \geq n} b_k$

$$\Rightarrow \inf_{k \geq n}(a_k b_k) \geq \left(\inf_{k \geq n} a_k\right)\left(\inf_{k \geq n} b_k\right)$$

$$\Rightarrow \underline{\lim}\,(a_n b_n) \geq \underline{\lim}\, a_n \, \underline{\lim}\, b_n$$

Power Series: An infinite series of the form $\sum a_n(z-a)^n$ is known as power series about a. (64)

The simplest power series is $\sum_{n=0}^{\infty} z^n$. We can show that

$$1-z^{n+1} = (1-z)(1+z+\cdots+z^n)$$

& $$1+z+\cdots+z^n = \frac{1-z^{n+1}}{1-z}$$

If $|z|<1$, then $\lim z^n = 0$ &

$$|1+z+\cdots+z^n| < 1+|z|+\cdots+|z|^n < \frac{1}{1-|z|} < \infty$$

Thus the series $\sum z^n$ is abs. conv, and hence convergent and

$$1+z+\cdots+z^n = \frac{1-0}{1-z}$$

$$\therefore \left|1+z+\cdots+z^n - \frac{1}{1-z}\right| = \frac{|z|^{n+1}}{|1-z|} \to 0.$$

Thm: Let $0 \le R \le \infty$ be such that (65)

$\frac{1}{R} = \limsup |a_n|^{1/n}$. Then the power series $\sum a_n (z-a)^n$ is

(i) Conv. abs. for $|z-a| < R$

(ii) divergent for $|z-a| > R$.

(iii) If $0 < r < R$, then the series is unif. conv. for $|z-a| \le r$.

Moreover, the number R, $0 \le R \le \infty$ is the only number having property (i) & (ii).

Proof: Without loss of generality, we can assume that $a = 0$. If $|z| < R$, then $\exists$ $r > 0$ s.t. $|z| < r < R$.

i.e. $\limsup |a_n|^{1/n} = \frac{1}{R} < \frac{1}{r}$.

By def^n of limit, $\exists$ $N \in \mathbb{N}$ s.t.

$\sup_{n \ge k} |a_n|^{1/n} < \frac{1}{r}$ for $k \ge N$

$\sup_{n \geq K} |a_n|^{1/n} < \frac{1}{r}$ for ...

$\Rightarrow |a_K|^{1/K} < \frac{1}{r}$ for $K \geq N$ (66)

$|a_K| < \frac{1}{r^K}$

$|a_K z^K| < \left(\frac{|z|}{r}\right)^K, \quad K \geq N$

$$\Rightarrow \sum_{K=N}^{\infty} |a_K z^K| \leq \sum_{K=N}^{\infty} \left(\frac{|z|}{r}\right)^K < \infty \text{ if } \left|\frac{z}{r}\right| < 1.$$

Thus, the power series converges for $|z| < R$.

Now, suppose $r < R$ & choose ρ s.t. $r < \rho < R$

As above $\exists\, N \in \mathbb{N}$ s.t.

$$|a_n| < \frac{1}{\rho^n}$$

$|z| \leq r \Rightarrow |a_n z^n| \leq \left(\frac{r}{\rho}\right)^n. \quad \frac{r}{\rho} < 1$

Hence by Weierstrass M-test, the power series conv. unif for $|z| < r < R$.

[Explanation: $\limsup |a_n|^{1/n} = \frac{1}{R}$ (C7)

$\left| \sup |a_n|^{1/n} - \frac{1}{R} \right| < \epsilon$ for large n.

$\sup |a_n|^{1/n} < \epsilon + \frac{1}{R} = \frac{1}{r} > \frac{1}{R}$

$\Rightarrow |a_n|^{1/n} < \frac{1}{r}$ for $n \geq N$.]

(ii) Let $|z| > R$. Choose $r > 0$ s.t. $|z| > r > R$.

$\frac{1}{r} < \frac{1}{R} = \limsup |a_n|^{1/n}$

$\Rightarrow \exists$ ∞ly many n st

$\frac{1}{r} < |a_n|^{1/n}$

$\Rightarrow |a_n z^n| > \left(\frac{|z|}{r} \right)^n \geq 1$

$\Rightarrow$ these terms are unbounded and hence the series is divergent.

[On contrary suppose $\frac{1}{r} < |a_n|^{1/n}$ (70)
for finitely many n, then

$$|a_n|^{1/n} \leq \frac{1}{r} \quad \text{for } n \geq N$$

$$\Rightarrow \sup_{k \geq n} |a_k|^{1/k} \leq \frac{1}{r} \quad \text{for } n \geq N$$

$$\Rightarrow \lim_{n \to \infty} \left(\sup_{k \geq n} |a_k|^{1/k} \right) \leq \frac{1}{r}$$

$$\Rightarrow \overline{\lim} |a_n|^{1/n} \leq \frac{1}{r}$$

is absurd.]

The number $R = \frac{1}{\limsup |a_n|^{1/n}}$ is known as radius of conv. of the power series $\sum a_n z^n$.

<u>Remark</u>: It may be possible that the power series $\sum a_n z^n$ may converge for some or every point of $|z| = R$, e.g. $\sum \frac{z^n}{n}$ converges for $z = -1$, and $\sum \frac{z^n}{n^2}$ converges for every point of $|z| = 1$

Proposition: If the power series $\sum a_n z^n$ has radius of conv. R, and $\lim \left|\frac{a_n}{a_{n+1}}\right|$ exists, then $R = \lim \left|\frac{a_n}{a_{n+1}}\right|$. (71)

Proof: Let $\alpha = \lim \left|\frac{a_n}{a_{n+1}}\right|$ & $|z| < \gamma < \alpha$.

Then $\exists$ $N \in \mathbb{N}$ s.t.

$$\gamma < \left|\frac{a_n}{a_{n+1}}\right| \quad \forall\, n \geq N$$

(by defn of limit).

Let $B = |a_N|\gamma^N$, then

$$|a_{N+1}|\gamma^{N+1} \leq |a_N|\gamma^N = B$$

$$\Rightarrow |a_n \gamma^n| \leq B \quad \forall\, n \geq N$$

But then

$$|a_n z^n| = |a_n \gamma^n| \left|\frac{z}{\gamma}\right|^n \leq B \frac{|z|^n}{\gamma}$$

Since $|z| < \gamma$, it follows that $\sum a_n z^n$ is absolutely conv. for $|z| < \gamma < \alpha$.

For every γ, $\Rightarrow$ $\alpha \leq R$.

On the other hand if $|z| > r > \alpha$, (72)

then $|a_n| < r|a_{n+1}|$ for n large $\geq N$

As before $|a_n r^n| \geq B = |a_N r^N|$, $\forall n \geq N$

$\Rightarrow |a_n z^n| \geq B\left|\frac{z}{r}\right|^n \to \infty$.

$\Rightarrow \sum a_n z^n$ is divergent for $r > \alpha$

$\Rightarrow R \leq \alpha \Rightarrow R = \alpha$.

Consider the series $\sum \frac{z^n}{n!}$, then.

$\lim \left|\frac{a_n}{a_{n+1}}\right| = \infty$

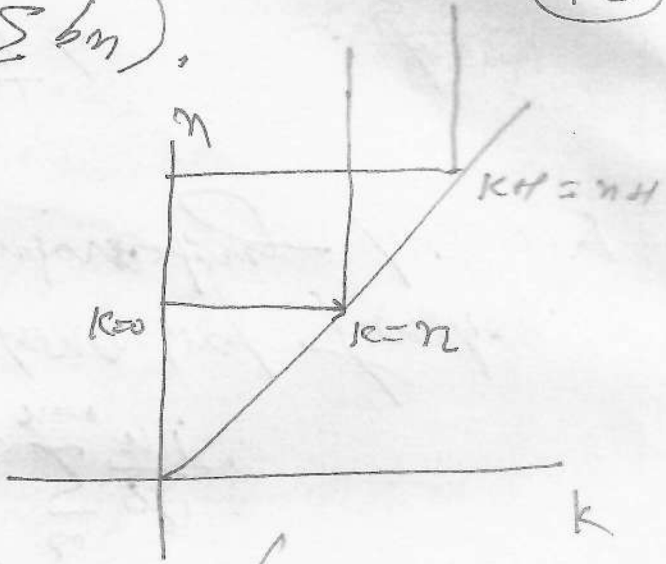
$\Rightarrow \sum \frac{z^n}{n!}$ Converges for every $z \in \mathbb{C}$.

Write $e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

This is known as exponential function, and derive many interesting properties very soon.

Prop: If $\sum a_n$ and $\sum b_n$ are two abs. conv. series, then, for $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, the series $\sum c_n$ is abs. conv. and

$$\sum c_n = (\sum a_n)(\sum b_n).$$



Proof: $\sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k||b_{n-k}|$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} |a_k||b_{n-k}|$$

$$= \sum_{k=0}^{\infty} |a_k| \sum_{n=k}^{\infty} |b_{n-k}|$$

$$= \left(\sum_{k=0}^{\infty} |a_k|\right)\left(\sum_{k=0}^{\infty} |b_k|\right) < \infty$$

(this is known as Fubini's thm of changing the order of sum)

Hence $\sum c_n$ is abs. conv. & by problem

$$\sum c_n = (\sum a_n)(\sum b_n).$$

Proof: Suppose $\sum a_n z^n$ & $\sum b_n z^n$ have r.c. $\geq r > 0$. Put $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then both the power series

$\sum (a_n+b_n)z^n$ & $\sum c_n z^n$ have radius of conv. $\geq r$, and

(74)

~~Proof~~: $\sum c_n z^n = \left(\sum a_n z^n\right)\left(\sum b_n z^n\right)$

proof: If $0 < s < r$, then $|z| < s < r$

$\Rightarrow \sum |a_n+b_n| |z|^n \leq \sum |a_n| s^n + \sum |b_n| s^n < \infty$

$\Rightarrow$ r.c. of $\sum (a_n+b_n) z^n \geq r$.

Also

$$\sum_{n=0}^{\infty} |c_n z^n| = \sum \sum |a_k| |b_{n-k}| s^k s^{n-k}$$

$$= \sum \sum \left(|a_k| s^k\right)\left(|b_{n-k}| s^{n-k}\right)$$

$$= \left(\sum |a_n| s^n\right)\left(\sum |b_n| s^n\right) < \infty$$

($\because$ abs. conv $\Rightarrow$ conv., the result follows, and r.c. of $\sum c_n z^n \geq r$.

Ex. Show that radius of conv. of $\sum (a_n+b_n) z$ is bigger than $\min(R_1, R_2)$, where

$\frac{1}{R_1} = \limsup |a_n|^{1/n}$, $\frac{1}{R_2} = \limsup |b_n|^{1/n}$.

75

Notice that $\frac{1}{R} = \limsup |a_n + b_n|^{1/n}$

$$\leq \limsup \{|a_n| + |b_n|\}^{1/n}$$

$$\leq \limsup \{2 \max\{|a_n|, |b_n|\}\}^{1/n}$$

$$= \max\left\{\frac{1}{R_1}, \frac{1}{R_2}\right\} = \frac{1}{\min(R_1, R_2)}$$

$\Rightarrow R \geq \min(R_1, R_2)$.

Remark: Suppose $R_1 \neq R_2$, then $R = \min\{R_1, R_2\}$.

$$\sum |(a_n + b_n) z^n| \leq \sum |a_n||z|^n + \sum |b_n||z|^n \quad (*)$$

If $R_1 < R_2$, then $(*)$ conv. for $|z| < r < R_1$

$\Rightarrow$ R.C. of $\sum (a_n + b_n) z^n$ $R \geq R_1$

On the other hand if $|z| = r > R_1$, then for $R_1 < r < R_2$, the series $\sum b_n z^n$ is conv.

$\Rightarrow R \leq R_1 \Rightarrow R = R_1 = \min(R_1, R_2)$

Note that $\limsup |a_n|^{1/n} > \limsup |b_n|^{1/n}$

$$> \liminf |b_n|^{1/n}$$

$\Rightarrow \limsup (|a_n|^{1/n} - |b_n|^{1/n}) > 0$.

$\Rightarrow a_n \neq b_n$ for ∞ly many n.

$\Rightarrow \ldots$ for any ...

Thm: If $f(z) = \sum a_n z^n$ has radius of conv. $R > 0$, then (a) for $k \geq 1$,

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n z^{n-k}$$

has radius of conv. R.

(76)

(b) f is infinitely diff. on $|z| < R$ and

$$f^{k}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n z^{n-k}$$

with $a_n = \dfrac{f^{n}(0)}{n!}$.

Proof: Consider $\sum_{n=1}^{\infty} n a_n z^{n-1}$ and $\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}$

Claim $\frac{1}{R} = \limsup |n a_n|^{\frac{1}{n-1}}$

Note that $\lim \frac{\log n}{n-1} = 0 \Rightarrow \lim n^{1/n-1} = 1$.

Let $\frac{1}{R'} = \limsup |a_n|^{\frac{1}{n-1}}$. Then R' is radius of conv. of

$$\sum_{n=1}^{\infty} a_n z^{n-1} = \sum_{n=0}^{\infty} a_{n+1} z^n$$

Note that

$$z\sum_{n=0}^{\infty} a_{n+1} z^{n} + a_0 = \sum a_n z^n. \quad \text{———} \; (*)$$

77

Hence if $|z|<R'$, then

$$\sum |a_n z^n| \le |a_0| + \sum_{n=0}^{\infty} |a_{n+1} z^n| < \infty$$

$$\Rightarrow R' \le R$$

Now, if $|z|<R$ & $z \neq 0$, then

$$\sum_{n=0}^{\infty} |a_{n+1} z^n| \le \frac{1}{|z|}\sum_{n=0}^{\infty} |a_n z^n| + \frac{1}{|z|}|a_n| < \infty$$

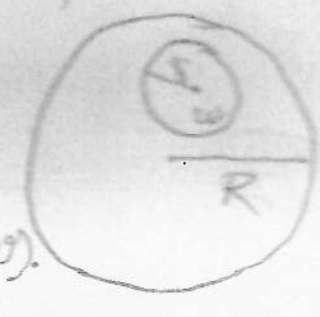
$\Rightarrow R' \ge R$. Hence $R' = R$.

(b) For $|z|<R$, let $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$,

$S_n(z) = \sum_{k=0}^{n} a_k z^k$, $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$.

Then $f = S_n + R_n$.

For $|w| < r < R$, we show that $f'(w)$ exists & $= g(w)$.



Let $\delta > 0$ s.t. $\overline{B(w,\delta)} \subset B(0,r)$.

Let $z \in B(w, \delta)$. then

$$\frac{f(z)-f(w)}{z-w} - g(w) = \left[\frac{S_n(z)-S_n(w)}{z-w} - S_n'(w)\right]$$

$$+\left[S_n'(w) - g(w)\right] + \left[\frac{R_n(z)-R_n(w)}{z-w}\right]$$

now,

$$\alpha_n(z) = \frac{R_n(z)-R_n(w)}{z-w} = \sum a_k\left(\frac{z^k-w^k}{z-w}\right)$$

By induction, it can be shown that

$$\frac{z^k-w^k}{z-w} = z^{k-1} + z^{k-2}w + \cdots + z w^{k-2} + w^{k-1}$$

(if $z \neq w$).

Since $z \in B(w,\delta) \subset B(0,r)$

$$\left|\frac{z^k-w^k}{z-w}\right| < k r^{n-1}$$

$$\Rightarrow |\alpha_n(z)| \le \sum_{k=n+1} |a_k| k r^{k-1} \longrightarrow 0 \text{ as } n \to \infty$$

($\because r < R$, the radius of conv. of $\sum n a_n z^{n-1}$ too).

For $\epsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $|\alpha_n(z)| < \epsilon/3$ $\forall n \geq N_1$ —(1)

Also, $S_n'(w) \to g(w)$. For $\epsilon > 0$, $\exists\ N_2 \in \mathbb{N}$

st $|S_n'(w) - g(w)| < \epsilon/3$ —— (2)

Now, choose $0 < \delta' < \delta^*$ st (79)

for $|z-w| < \delta'$ $\Rightarrow \left| \dfrac{S_n(z) - S_n(w)}{z-w} - S_n'(w) \right| < \epsilon/3$

for large $n \geq \max(N_1, N_2)$, we —— (3)

get $\left| \dfrac{f(z) - f(w)}{z-w} - g(w) \right| < \epsilon$ for $|z-w| < \delta'$.

$\Rightarrow f'(w) = g(w) = \sum\limits_{k=1}^{\infty} a_k k\, w^{k-1}$

$\Rightarrow f'(0) = a_1$ & $f(0) = a_0$.

Similarly, if we write

$g(z) = \sum\limits_{k=n}^{\infty} n(n-1)\cdots(n-k+1)\, a_k z^{n-k}$

$f^{n}(0) = g(0) = n(n-1)\cdots(n-n+1)\, a_n$

$\Leftrightarrow\ a_n = \dfrac{f^{n}(0)}{n!}$.

$n!$

Cor: If $\sum a_n z^n$ has radius of conv R, then $f(z) = \sum a_n z^n$ is analytic in $B(0,R)$.

80

Hence $\exp(z) := \sum \frac{z^n}{n!}$ is analytic in $\mathbb{C}$.

Note that $f(z) = e^z$ is analytic, and

$$f'(z) = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z.$$

$$\Rightarrow \frac{d}{dz} e^z = e^z.$$

Put $g(z) = e^z e^{a-z}$ for fixed a, then

$$g'(z) = e^z e^{a-z}(-1) + e^z e^{a-z} = 0$$

$$\Rightarrow g(z) = w \text{ (const)}$$

$$\Rightarrow w = g(0) = e^a$$

$$\Rightarrow e^a = e^z e^{a-z}$$

Put $a \to a+b$, & $z \to b$, then

$$e^a e^b = e^{a+b}, \ \forall a, b \in \mathbb{C}$$ (81)

$$\Rightarrow e^z e^{-z} = e^0 = 1$$

$$\Rightarrow e^z \neq 0 \ \& \ e^{-z} = \frac{1}{e^z}.$$

(i.e. each of them is inverse of other).

Notice that $\overline{\exp(z)} = \exp(\bar{z})$, as each coeff. of e^z is real.

If θ is real, then

$$|e^{i\theta}|^2 = e^{i\theta}\overline{e^{i\theta}} = e^{i\theta} \cdot e^{-i\theta} = e^0 = 1.$$

$$\Rightarrow |e^{i\theta}| = 1.$$

$$|e^z|^2 = e^z e^{\bar{z}} = \exp(2\operatorname{Re} z)$$

$$\Rightarrow |e^z| = \exp(\operatorname{Re} z)$$

Now, $e^z = e^{x+iy} = e^x e^{iy}$

$|e^z| = \exp(\operatorname{Re} z)$, $\arg(e^z) = \operatorname{Im} z$.

Now, we define

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{(-1)^n z^{2n}}{(2n)!} + \cdots$$

82

$$\& \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \cdots$$

Both of them of radius of conve ∞.

Hence $\cos z$ & $\sin z$ are entire.

By previous result, we can deduce that

$$(\cos z)' = -\sin z$$

$$\& \ (\sin z)' = \cos z$$

Also, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$$\& \ \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$\Rightarrow \cos^2 z + \sin^2 z = 1 \text{ for } z \in \mathbb{C}$$

$$\& \ e^{iz} = \cos z + i \sin z.$$

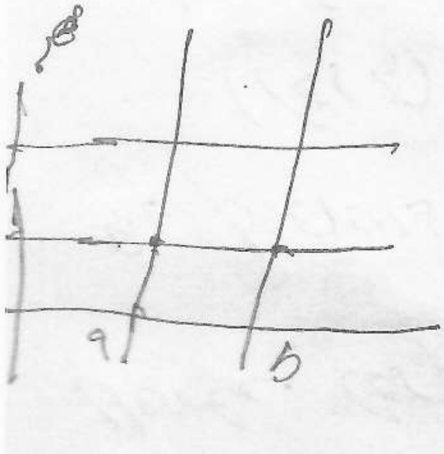
In particular, if $z = \theta$ real, then

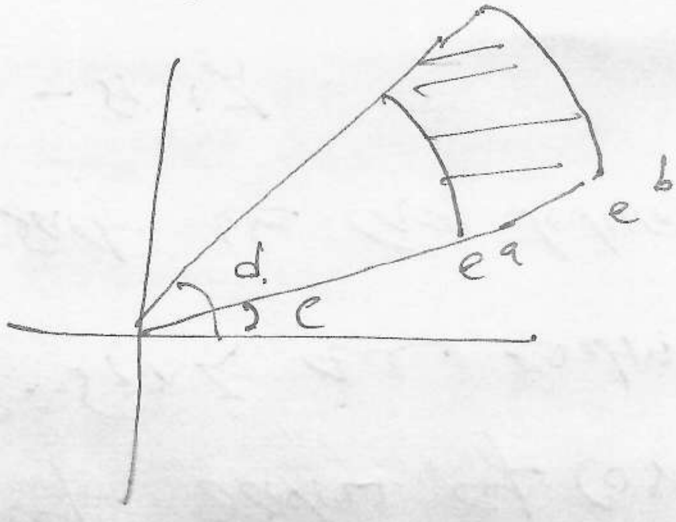
Then we get $e^{i\theta} = \cos\theta + i\sin\theta$ (83)
(The Euler's formula).

<u>Mapping property of exponential.</u>

$\{(x, y_0) : x \in \mathbb{R}\} \longrightarrow \{e^{x} e^{iy_0} : x \in \mathbb{R}\}$

$\{(x_0, y) : y \in \mathbb{R}\} \longrightarrow \{e^{x_0} e^{iy} : y \in \mathbb{R}\}$





Ex. Show that $\sin z$ is unbounded on $\mathbb{C}$.

$$|\sin z| = \frac{|e^{ix}e^{-y} - e^{-ix}e^{y}|}{2} \to \infty$$

as $|y| \to \infty$.

Complex logarithm

Now, we establish a relation between exp. f^n and its (inverse) logarithmic function.

For this, consider $e^{w} = z$, for $z \neq 0$, $w \in \mathbb{C}$.

Notice that e^z is not injective function on $\mathbb{C}$ because

$$e^{z+2k\pi i} = e^{z} \text{ for } k \in \mathbb{Z}.$$

However, e^z is an onto function from $\mathbb{C}$ to $\mathbb{C}\setminus\{0\} = \mathbb{C}^*$.

For $w \in \mathbb{C}^*$, $w = |w| e^{i\theta}$, for some $\theta \in (-\pi, \pi]$.

Consider $z = \log|w| + i\theta$, then

$$e^{z} = e^{\log|w| + i\theta} = |w| e^{i\theta} = w$$

It is clear from the above that if we restrict domain of exp to

$$H = \{ z = x+iy : x \in \mathbb{R},\ y \in (-\pi, \pi] \}$$

then exp is one-one & onto.

i.e. $\exp : H \longrightarrow \mathbb{C}^*$ is a bijection (85)
and ~~and~~ hence we can think of its inverse.

Notice that $\log z = \ln|z| + i(\theta + 2k\pi)$, $k \in \mathbb{Z}$.

We denote the principle value of logarithm by Log and

$$\mathrm{Log} : \mathbb{C}^* \longrightarrow H \text{ by}$$

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z) \text{ ; where}$$

$$\mathrm{Arg}(z) \in (-\pi, \pi].$$

If $z \neq 0$, then
$$e^{\mathrm{Log}\, z} = e^{\ln|z| + i\,\mathrm{Arg}(z)}$$
$$= |z|\, e^{i\,\mathrm{Arg}(z)}$$
$$= z.$$

On the other hand, we may be curious to know, what is the value of $\mathrm{Log}(e^z)$?

Note that $\mathrm{Log} : \mathbb{C}^* \longrightarrow H$

& $\exp : H \longrightarrow \mathbb{C}^*$

Hence Log is not cont. on $(-\infty, 0]$, but Log is analytic on $\mathbb{C} \setminus \{(-\infty, 0]\}$. (87)

Note that $\operatorname{Log}(e^z) = \ln|e^z| + i\operatorname{Arg}(e^z)$

iff $\operatorname{Arg}(e^z) \in (-\pi, \pi]$

iff $\operatorname{Arg}(e^x e^{iy}) \in (-\pi, \pi]$

iff $y \in (-\pi, \pi]$

Thus $\operatorname{Log}(e^z) = z$ iff $\operatorname{Im}(z) \in (-\pi, \pi]$.

For $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(r_1 r_2 e^{i(\theta_1 + \theta_2)})$

$= \ln r_1 r_2 + i(\theta_1 + \theta_2)$

iff $\theta_1 + \theta_2 \in (-\pi, \pi]$

$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$ iff

$\operatorname{Arg}(z_1 + z_2) \in (-\pi, \pi]$.

$\operatorname{Log}(z^2) = 2\operatorname{Log}(z)$ iff $\operatorname{Arg}(z) \in (-\pi/2, \pi/2]$.

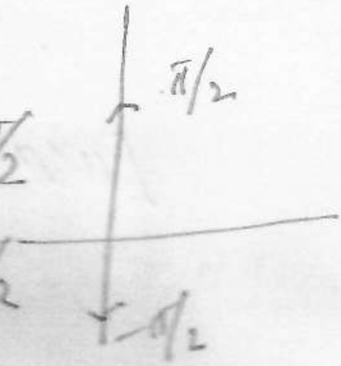
So $\mathrm{Log}(e^{z}) = \mathrm{Log}(e^{x} e^{iy})$

$= \ln|e^{x}e^{iy}| + i\,\mathrm{Arg}(e^{x}e^{iy})$

$= \ln e^{x} + iy$

$= x+iy = z, \quad z \in H$

ie $\mathrm{Log} \circ \exp : H \longrightarrow H$.



Ex. $\mathrm{Log}(i) = \ln|i| + i\pi/2 = i\pi/2$

$\mathrm{Log}(-i) = \ln|-i| + i(-\pi/2) = -i\pi/2$

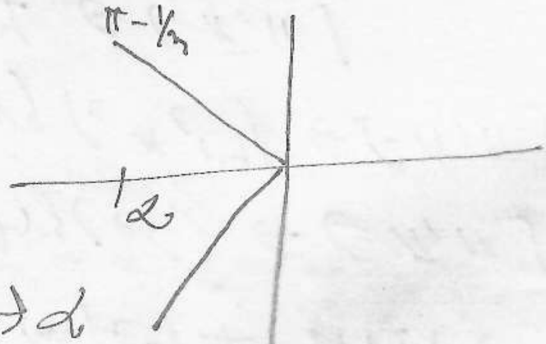
$\mathrm{Log}(-e) = \ln|-e| + \mathrm{Arg}(-e)$

$= 1 + i\pi$

Remark: the function Log is not cont. on negative real axis. $(-\infty, 0]$.

$a_n = \alpha\, e^{i(\pi - 1/n)}$

& $b_n = \alpha\, e^{i(-\pi + 1/n)}$



Then $a_n \to \alpha$ & $b_n \to \alpha$

But $\mathrm{Log}\, a_n = \ln|\alpha| + i(\pi - \frac{1}{n}) \to \ln|\alpha| + i\pi$

$\mathrm{Log}\, b_n = \ln|\alpha| + i(-\pi + 1/n) \to \ln|\alpha| - i\pi$

Now we partially examine for when

$$z^w = \exp(w \operatorname{Log}(z))?$$

(98)

Suppose $w = \alpha \in \mathbb{R}_+ = (0, \infty)$. Then

claim $\operatorname{Log} z^\alpha = \alpha \operatorname{Log}(z)$

let $z = re^{i\theta}$, $\theta \in (-\pi, \pi]$

$LHS = \operatorname{Log}(r^\alpha e^{i\alpha\theta})$

$= \alpha \log r + i\alpha\theta$ iff $\alpha\theta \in (-\pi, \pi]$

$= \alpha(\log r + i\theta)$ iff $\theta \in (-\pi/\alpha, \pi/\alpha]$

$= \alpha \operatorname{Log}(z)$ iff $\operatorname{Arg}(z) \in (-\pi/\alpha, \pi/\alpha]$

$= RHS$ iff $\operatorname{Arg}(z) (-\pi, \pi]$

A partial solution for α is $\alpha \in (0, 1)$.

That is $\operatorname{Log} z^\alpha = \alpha \operatorname{Log} z$ if $\alpha \in (0,1)$.

If we write

$$f(w) = z^w = \exp(w \operatorname{Log} z)$$

then f is analytic on $\mathbb{C} \setminus (-\infty, 0]$

and $f(w) = 0$ for $w \in (0,1)$.

we show later that $f \equiv 0$. (??).

Ex $i^i = \exp(i \operatorname{Log} i)$ 89

$$= \exp(i \ln|i| + i^2 \operatorname{Arg} i)$$

$$= \exp(0 + (-1)\pi/2) = \exp(-\pi/2).$$

Ex. Show that

$$\operatorname{Log} z^\alpha = \alpha \operatorname{Log}(z) \quad \text{for } \alpha \in (-1,0).$$

(Hint: $\alpha = -\beta,\ \beta > 0;$

$$\operatorname{Log} \frac{1}{z^\beta} = \beta \operatorname{Log} \frac{1}{z} \text{ etc})$$

Inverse of function $w = z^n$

Let $w = |w| e^{i\theta_0}$, then

$$z = f(w) = |w|^{1/n} e^{i\frac{(\theta_0 + 2k\pi)}{n}} ; \quad k = 0, 1, \dots, n-1$$

That is, for given w, z will takes n many values. The ~~function~~ symbol F is known as multivalued function.

For the case $n = 2$, $z^2 = w$

$$\Rightarrow z = |w|^{1/2} e^{\frac{i(\theta + 2k\pi)}{2}} ; k = 0, 1$$

$$z = \begin{cases} |w|^{1/2} e^{i\theta/2} \\ |w|^{1/2} e^{i(\theta/2 + \pi)} \end{cases}$$

$$= \begin{cases} +|w|^{1/2} e^{i\theta/2} \\ -|w|^{1/2} e^{i\theta/2} \end{cases} = \{ \pm\sqrt{w}$$

That is $z = F(w) = \pm\sqrt{w}$

$\Rightarrow F$ is two-valued function.

In order to discuss such multi-valued function, we need to specify the domain of each different value.

<u>Branch of multi-valued functions</u>

Let F be a multi-valued function on domain D. A function f is said to be branch of F if $\exists$ a subdomain $D_f \subset D$ st $F/_{D_f} = f$ is a single-valued analytic function.

That is, every single-valued analytic restriction of F to a smaller domain is a branch of F. (91)

For example $z^2 = w$ has two branches

$$z = f_1(w) = |w|^{1/2} e^{i\theta/2} \quad \theta \in (-\pi, \pi]$$

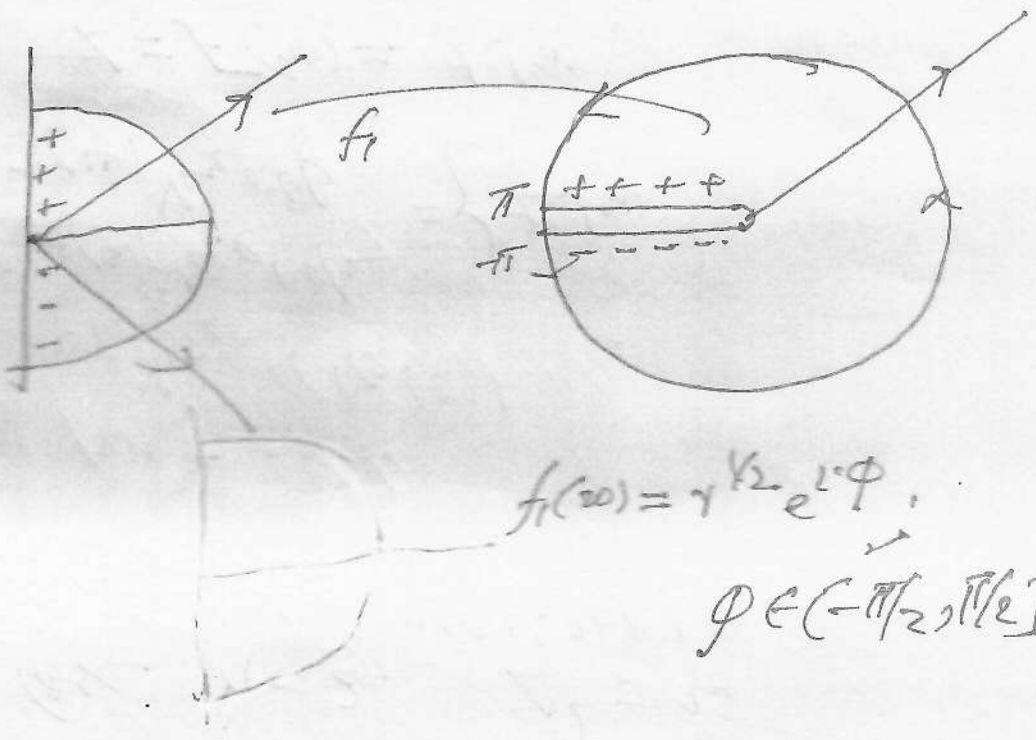
$$\text{and } z = f_2(w) = |w|^{1/2} e^{i(\theta/2 + \pi)} \quad \theta \in (-\pi, \pi]$$

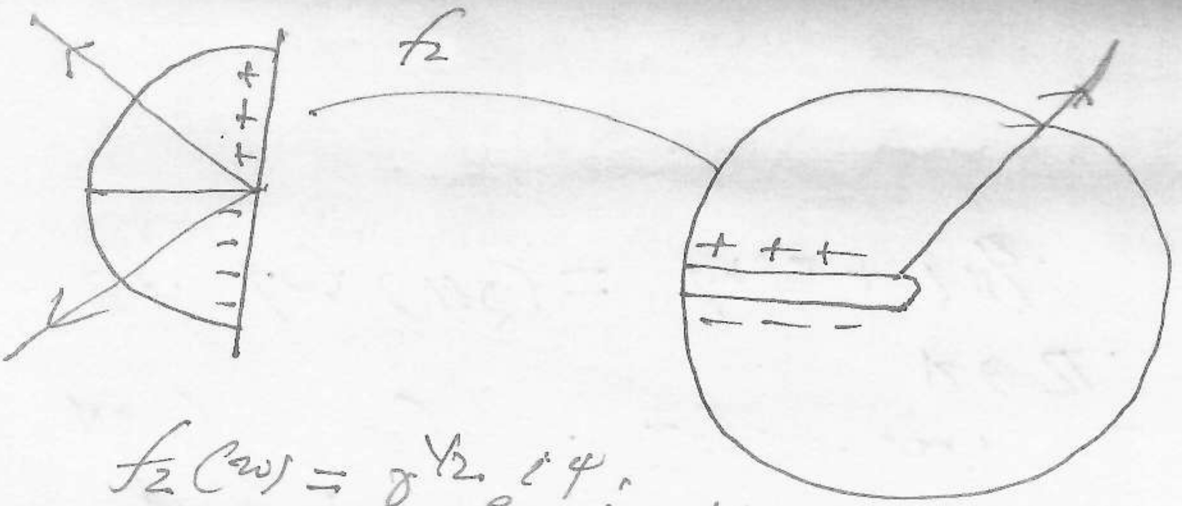
$$\text{or } f_1(w) = |w|^{1/2} e^{i\theta/2} \; ; \; \theta/2 \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\& \; f_2(w) = |w|^{1/2} e^{i(\theta/2 + \pi)} \; ; \; \pi + \theta/2 \in (\pi/2, 3\pi/2]$$

$$\text{i.e. } f_1(w) = |w|^{1/2} e^{i\varphi} \; ; \; \varphi \in (-\pi/2, \pi/2]$$

$$\& \; f_2(w) = |w|^{1/2} e^{i\varphi} \; ; \; \varphi \in (\pi/2, 3\pi/2].$$





$f_2(w) = r^{1/2} e^{i\psi}$; $\psi \in (+\pi/2, 3\pi/2]$

That is, f_2 maps $\mathbb{C} \setminus (-\infty, 0]$ to left half plane and f_1 to right half plane.

This gives idea to define surface by gluing these two images where f_1 & f_2 coincides. The resultant surface is known as Riemannian surface of multi-valued function F.

In the above construction $(-\infty, 0]$ is known as branch cut and 0 as branch point.

In general, a branch cut is a portion of a line or curve introduced to define a branch, whereas branch point ∞ common pt all branches.

(23)

<u>Branch of complex logarithm</u>:

For $z \neq 0$, consider $e^w = z$ —— (*)

Then $z = \log|z| + i \arg(w)$ is a solution of (*).

Recall that the principle value of Log z is given by

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg}(z), \text{ with}$$
$$\operatorname{Arg}(z) \in (-\pi, \pi]$$

Hence Log z is a single valued image of e^w.

Now, $\log z = \operatorname{Log} z + 2k\pi i$ —— (*a)
$k \in \mathbb{Z}$.

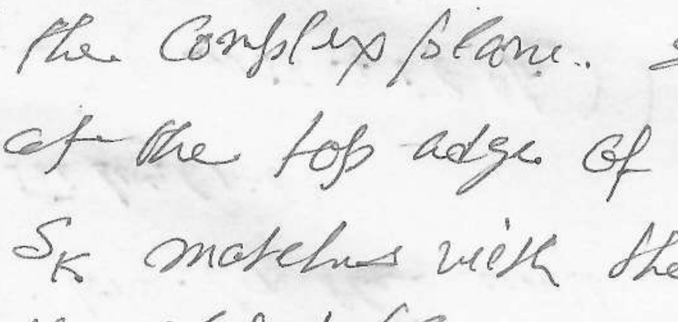
Ex. $\operatorname{Log}(1+i) = \ln\sqrt{2} + i\pi/2$

(25)

Recall that $w = \operatorname{Log} z$ is an analytic function on $\mathbb{C} \setminus (-\infty, 0]$. From (**), it is clear that there are infinitely many branches of $z = e^{w}$.

Let $f_K(w) = \log w + 2K\pi i$; $K \in \mathbb{Z}$

For each branch say f_K, we take a copy of the complex plane and slit it along the real axis to obtain a copy S_K of the plane $\mathbb{C} \setminus (-\infty, 0]$. We write f_K as a function on the kth sheet S_K of the complex plane. Since value of f_K at the top edge of the slitted plane S_K matches with the bottom of edge of the slitted plane S_{K+1}, we glue these edges together and continue the process to obtain a Riemannian surface.



Hence we get an infinite spiral about the origin. (95)

Ex. Draw the branches of $z = e^{1+w}$.

Ex. Show that all the values of $(1-i)^{\sqrt{2}i}$ lies on a straight line in the complex plane.

$$(1-i)^{\sqrt{2}i} = e^{\sqrt{2}i \log(1-i)}$$

$$= e^{\sqrt{2}i\left(\ln\sqrt{2} + i\left(-\pi/2\right)\right)}$$

$$= e^{\sqrt{2}\ln\sqrt{2}\, i + \sqrt{2}\,\pi/2} \text{ etc.}$$

Ex. Show that $\log\left(\frac{1}{z}\right) = -\log z$ for all $z \in \mathbb{C} \setminus \{0\}$.