

# Complex Integration

96

Recall that

$f: [a, b] \xrightarrow{\text{cont}} \mathbb{R}$  is  $\mathbb{R}$ -integrable.

i.e. for  $\epsilon > 0$ ,  $\exists \delta > 0$  st

$$\|P\| = \max \{ \Delta t_j ; \Delta t_j = t_j - t_{j-1} ; j = 1, 2, \dots, n \} < \delta$$

$$\Rightarrow U(P, f) - L(P, f) < \epsilon$$

$$\text{Equivalently, } \sup L(P, f) = \inf U(P, f) = \int_a^b f(t) dt$$

$\Leftrightarrow$  for every increasing seq<sup>n</sup>  $P_n$  of partitions of  $[a, b]$  (denoted by  $\mathcal{P}[a, b]$ ),

$$\lim U(P_n, f) = \lim L(P_n, f) = \int_a^b f(t) dt$$

$$\text{Hence, } U(P_n, f) = \sum m_j \Delta t_j ; m_j = \sup_{t_{j-1} \leq t \leq t_j} f(t)$$

$$\wedge L(P_n, f) = \sum m_j \Delta t_j ; m_j = \inf_{t_{j-1} \leq t \leq t_j} f(t)$$

The space of all Riemann integrable functions is denoted by  $R[a, b]$ .

Let Partition  $P_n = \{a = t_0 < \dots < t_n = b\}$ , write (97)

$$S(P_n, f, \{\xi_j\}) = \sum_{j=1}^n f(\xi_j) \Delta t_j, \quad \xi_j \text{ is any point in } [t_{j-1}, t_j]$$

it can be shown that

$$S(P_n, f, \{\xi_j\}) \rightarrow \int_a^b f(x) dx.$$

Thm:  $f \in R[a, b]$  iff  $S(P_n, f, \{\xi_j\}) \rightarrow I$ ,  
where  $I = \int_a^b f(x) dx$ .

Proof: Suppose  $f \in R[a, b]$ , then

$$m_j \leq f(\xi_j) \leq M_j$$

$$\Rightarrow \sum_{j=1}^n \Delta t_j m_j \leq \sum_{j=1}^n f(\xi_j) \Delta t_j \leq \sum_{j=1}^n M_j \Delta t_j$$

$$\Rightarrow L(P_n, f) \leq S(P_n, f, \{\xi_j\}) \leq U(P_n, f)$$

$$\text{Since } \lim L(P_n, f) = \lim U(P_n, f)$$

$$\Rightarrow \lim S(P_n, f, \{\xi_j\}) = \int_a^b f(x) dx$$

Conversely, suppose  $\lim S(P_n, f, \{\xi_j\}) = I$

We claim that  $f \in R[a, b]$  &  $I = \int_a^b f(x) dx$ . (98)

For  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$  st

$$n \geq N \Rightarrow I - \epsilon/4 < S(P_n, f, \{\xi_j\}) < I + \epsilon/4$$

In particular

$$I - \epsilon/4 < S(P_n, f, \{\xi_j\}) < I + \epsilon/4$$

for any point  $\xi_j \in [t_{j-1}, t_j]$ . — (1)

$$\text{Let } m_j = \inf_{t_{j-1} \leq t \leq t_j} f(t), \quad M_j = \sup_{t_{j-1} \leq t \leq t_j} f(t)$$

Then  $\exists \alpha_j, \beta_j \in [t_{j-1}, t_j]$  s.t.

$$m_j - \frac{\epsilon}{4(b-a)} < m_j < f(\alpha_j) < m_j + \frac{\epsilon}{4(b-a)} \quad (2)$$

$$M_j - \frac{\epsilon}{4(b-a)} < f(\beta_j) < M_j + \frac{\epsilon}{4(b-a)} \quad (3)$$

Multiplying (2) & (3) with  $\Delta t_j$  & summing  
and combining with (1), we get

$$U(P_n, f) - L(P_n, f) < \epsilon.$$

$$\therefore I = \int_a^b f(x) dx.$$

Def<sup>n</sup>: A curve  $\gamma: [a, b] \rightarrow \mathbb{C}$  is said (99)  
to be smooth if  $\gamma'$  is continuous. And  
similarly piecewise smooth.

$$\text{ie } \gamma = \sum_{j=1}^n \gamma_j|_{[t_{j-1}, t_j]}, \text{ where}$$

each  $\gamma_j$  is smooth.

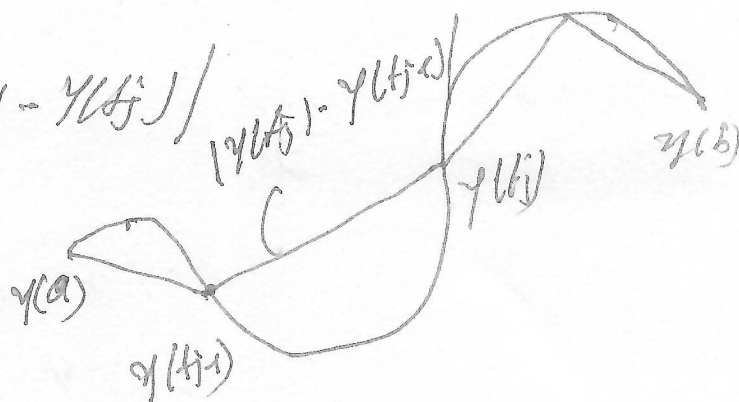
For partition  $P = \{a = t_0, \dots, t_{j-1}, t_j, \dots, t_n = b\}$ ,

write.

$$V(\gamma, P) = \sum_{j=1}^n |\gamma(t_{j-1}) - \gamma(t_j)|$$

If  $\exists M > 0$  st

$$V(\gamma, P) \leq M < \infty$$



for any partition  $P$  of  $[a, b]$ , then we  
say  $\gamma$  is of bounded variation on  $[a, b]$ .

In this case, we write

$$V(\gamma) = \sup_P V(\gamma, P) \leq M < \infty.$$

We claim that  $V(\gamma) = \int_a^b |\gamma'(t)| dt$ .



Ex. Show that  $\gamma$  is of bounded variation iff  $Re(\gamma)$  &  $Im(\gamma)$  both are of bounded variation. (100)

Ex. If  $P \subset Q$ , then show that

$$V(\gamma, P) \leq V(\gamma, Q).$$

Proof: Suppose  $\gamma: [a, b] \rightarrow \mathbb{C}$  is piecewise smooth, then  $\gamma$  is of bounded variation &

$$V(\gamma) = \int_a^b |\gamma'(t)| dt.$$

Proof: w.l.o.g., assume  $\gamma$  is smooth

&  $P = \{a_0 = a < \dots < t_{j-1} < t_j < \dots < t_n = b\}$  is partition.

$$\begin{aligned} V(\gamma, P) &= \sum |\gamma(t_j) - \gamma(t_{j-1})| \\ &= \sum \left| \int_{t_{j-1}}^{t_j} \gamma'(t) dt \right| \leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |\gamma'(t)| dt \\ &= \int_a^b |\gamma'(t)| dt. \end{aligned}$$

$$\Rightarrow V(\gamma) \leq \int_a^b |\gamma'(t)| dt < \infty$$

$\Rightarrow \gamma$  is of b.v.

On the other hand,  $\gamma'$  is uniformly cont. on  $[a, b]$ . For  $\epsilon > 0$ ,  $\exists \delta_1 > 0$  st

$$|s - t| < \delta_1 \Rightarrow |\gamma'(s) - \gamma'(t)| < \epsilon.$$

Also,  $\gamma'$  is R-integrable. For  $\epsilon > 0$ ,  $\exists \delta_2 > 0$  st for  $\|P\| < \delta_2$

$$\left| \int_a^b |\gamma'(t)| dt - \sum |\gamma'(\xi_j)| \Delta t_j \right| < \epsilon$$

where  $\xi_j$  is any pt on  $[t_{j-1}, t_j]$

$$\Rightarrow \int_a^b |\gamma'(t)| dt < \epsilon + \sum_{j=1}^n |\gamma'(\xi_j)| \Delta t_j$$

$$= \epsilon + \sum_{j=1}^n \left| \int_{t_{j-1}}^{t_j} \gamma'(\xi_j) dt \right|$$

$$< \epsilon + \sum_{j=1}^n \left| \int_{t_{j-1}}^{t_j} (\gamma'(\xi_j) - \gamma'(t)) dt \right| + \sum_{j=1}^n \left| \int_{t_{j-1}}^{t_j} \gamma'(t) dt \right|$$

Let  $\delta = \min\{\delta_1, \delta_2\}$  then

$$\leq \epsilon + \epsilon(b-a) + \sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})|$$

(102)

$$\leq (1+b-a)\epsilon + V(\gamma), \quad \forall \epsilon > 0$$

$$\Rightarrow \int_a^b |\gamma'(t)| dt \leq V(\gamma)$$

(Note that we used the fact that (\*) holds for every tag.)

Thm: Let  $\gamma: [a,b] \rightarrow \mathbb{C}$  be of B.V.  
and  $f: [a,b] \rightarrow \mathbb{C}$  cont, then  $\exists I \in \mathbb{C}$   
st.  $\forall \epsilon > 0, \exists \delta > 0$  st.

$$|P| < \delta \Rightarrow |I - \sum_{k=1}^n f(t_k)(\gamma(t_k) - \gamma(t_{k-1}))| < \epsilon$$

In this case we denote  $I$  by

$$I = \int_a^b f d\gamma = \int_a^b f(t) d\gamma(t).$$

In addition, if  $\gamma$  is smooth, then  $I = \int_a^b f(t) \gamma'(t) dt$

Proof: note that  $f$  is unif. cont on (103)

[a,b]. we can find  $\delta_1 > \delta_2 > \dots > \delta_m$

st.  $|s-t| < \delta_m \Rightarrow |f(s) - f(t)| < \frac{1}{m}$

for  $m \geq 1$ , let  $P_m =$  collection of all partitions  $P$  with  $|P| < \delta_m$ .

Then  $P_1 \supset P_2 \supset \dots \supset P_m \dots$

write  $S(P) = \sum_{k=1}^n f(t_k)(y(t_k) - y(t_{k-1}))$ .

let  $F_m = \overline{\{S(P) : P \in P_m\}} = \overline{E_m}$ .

then following will be true

(\*)  $\left\{ \begin{array}{l} F \supset F_2 \supset \dots \text{ and} \\ \text{diam } F_m \leq \frac{2}{m} V(Y). \end{array} \right.$

If this is true, then by Cantor's intersection

theorem,  $\exists ! I \in \mathbb{R}$  st.  $I \in F_m \forall m \geq 1$



let  $\epsilon > 0$ , and consider  $m > \frac{2}{\epsilon} V(Y)$ . Then (104)

$$\epsilon > \frac{2}{m} V(Y) > \text{diam } F_m.$$

Since  $I \in F_m \neq m \cap I$ ,  $F_m \in B(I, \epsilon)$

for  $m \geq m_0(I)$ .

Then if  $S = S_{m_0}$ , then

$$\Rightarrow |I - S(P)| < \epsilon$$

for  $|P| < \delta$ . This proves the theorem.

Now, prove (\*):

$F \supset F_2) \dots$  is finite.

We claim diameter of  $F_m \leq \frac{2}{m} V(Y)$ .

For  $m \geq 1$ , let  $p \in P_m$  &  $q \in Q$ ,  
then  $q \in P_m$ .

claim  $|S(p) - S(q)| < \frac{1}{m} V(Y)$ .

Let  $1 \leq p \leq m$  &  $t_{p-1} < t^* < t_p$

(1.5)

put  $Q = P \cup \{t^*\}$ . If  $t_{p-1} \leq \sigma \leq t_p$ ,  
 $t^* \leq \sigma \leq t_p$

$$S(Q) = \sum_{k \neq p} f(t_k) [\gamma(t_k) - \gamma(t_{k+1})] \\
+ f(\sigma) [\gamma(t^*) - \gamma(t_{p-1})] \\
+ f(\sigma) [\gamma(t_p) - \gamma(t^*)]$$

By using the fact that

$$|f(\sigma) - f(t)| \leq \frac{1}{m} \text{ if } |\sigma - t| < \delta_m$$

$$|S(P) - S(Q)| = \left| \sum_{k \neq p} [f(t_k) - f(\sigma)] [\gamma(t_k) - \gamma(t_{k+1})] \right.$$

$$+ f(t_p) [\gamma(t_p) - \gamma(t_{p+1})]$$

$$- f(\sigma) [\gamma(t^*) - \gamma(t_{p-1})]$$

$$- f(\sigma) [\gamma(t_p) - \gamma(t^*)]$$

$$\leq \frac{1}{m} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k+1})| + |f(t_p) - f(\sigma)|$$

$$\leq \frac{1}{n} \sum_{k \neq p} |\gamma(t_k) - \gamma(t_{k+1})| + \frac{1}{n} |\gamma(t^*) - \gamma(t_{p+1})| + \frac{1}{n} |\gamma(t_p) - \gamma(t^*)|$$

(166)

$$\leq \frac{1}{n} V(\gamma).$$

Finally, let  $P, R \in P_n$  & write

$$Q = PUR.$$

$$|S(P) - S(R)| \leq |S(P) - S(Q)| + |S(Q) - S(R)|$$

$$\leq \frac{2}{n} V(\gamma).$$

Theorem. If  $\gamma: [a, b] \rightarrow \mathbb{C}$  smooth and  $f: [a, b] \rightarrow \mathbb{C}$  cont, then

$$\int_a^b f \, d\gamma = \int_a^b f(t) \gamma'(t) dt$$

Proof: By going through real and

imaginary parts of  $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ ,  
 we may assume that  $\text{Re } \gamma \in R$ . (107)

For  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t.

$$\|P\|K\delta \Rightarrow \left| \int_a^b f d\gamma - \sum f(\xi_j) [\gamma(t_j) - \gamma(t_{j-1})] \right| < \epsilon \quad (1)$$

By MVT, we get

$$\left| \int_a^b f d\gamma - \sum f(\xi_j) \gamma'(\xi_j') \Delta t_j \right| < \epsilon \quad (2)$$

Since (1) is true for any tag  $\xi_j \in [t_{j-1}, t_j]$ ,

$$(2) \Rightarrow \left| \int_a^b f d\gamma - \sum f(\xi_j') \gamma'(\xi_j') \Delta t_j \right| < \epsilon$$

$$\text{Also } \left| \int_a^b f(t) \gamma'(t) dt - \sum f(\xi_j) \gamma'(\xi_j) \Delta t_j \right| < \epsilon$$

$$\Rightarrow \left| \int_a^b f d\gamma - \int_a^b f(t) \gamma'(t) dt \right| < \epsilon \quad \forall \epsilon > 0$$

$$\Rightarrow \int_a^b f d\gamma = \int_a^b f(t) \gamma'(t) dt$$

Next, we show that integral along a  
 rectifiable curve can be approximated by



# polygon path

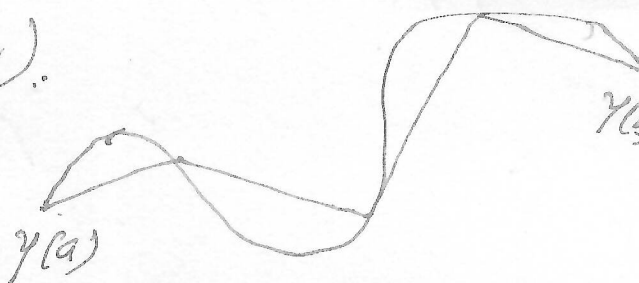
108

Prop: Let  $G$  be an open set in  $\mathbb{C}$  &  
 $\gamma: [a, b] \rightarrow G$  be a rectifiable path  
 &  $f: G \rightarrow \mathbb{C}$  be continuous. Then  
 $\forall \epsilon > 0$ ,  $\exists$  a polygon  $P$  in  $G$  st  
 $P(a) = \gamma(a)$ ,  $P(b) = \gamma(b)$  &

$$\left| \int_{\gamma} f dz - \int_P f dz \right| < \epsilon.$$

Proof: Suppose  $G = B(a, r)$ .

Since image of  $\gamma$   
 is compact,



$$d = \text{dist}(\text{Image}(\gamma), \partial G) > 0.$$

Hence, we can put

$$\text{Image}(\gamma) \subset B(a, f),$$

$$\text{where } f = r - \frac{1}{2}d.$$

Thus,  $\overline{B}(a, f) \subset G \Rightarrow f$  is contin. const  
 on  $\overline{B}(a, f)$ . Hence, wlog, we can

Assume that  $f$  is unif. cont. on  $G$ . (109)

Then for  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t.

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \quad \text{--- (1)}$$

Also,  $\gamma: [a, b] \rightarrow G$  is unif. cont.

So there is a partition  $P = \{t_0, t_1, \dots, t_n\}$  of  $[a, b]$  with  $\|P\| < \delta$  &

$$|\gamma(t_i) - \gamma(t_{i+1})| < \epsilon$$

for any  $t_i \in [t_i, t_{i+1}]$ , and. --- (2)

$$\left| \int_{\gamma} f - \sum f(\gamma(t_i)) [\gamma(t_i) - \gamma(t_{i+1})] \right| < \epsilon \quad \text{--- (3)}$$

Define  $p(t) = \frac{1}{t_i - t_{i+1}} [(t_i - t)\gamma(t_{i+1}) + (t - t_{i+1})\gamma(t_i)]$   
if  $t \in [t_i, t_{i+1}]$ .

Then  $p(t)$  is a line segment from  $\gamma(t_{i+1})$  to  $\gamma(t_i)$ . That is,  $P$  is a polygon in  $G$ .

From (2), it follows that

$$|p(t) - \gamma(t_i)| < \epsilon, \quad t_i < t < t_{i+1}. \quad \text{--- (4)}$$

Since  $\int_P f = \int_a^b f(P(t)) P'(t) dt$ , we get

$$\int_P f = \sum \frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} f \circ P(t) dt$$

(110)

Using (3), we obtain

$$\begin{aligned} \left| \int_P f - \int_P f \right| &< \epsilon + \left| \sum f(\gamma(t_j)) [\gamma(t_j) - \gamma(t_{j-1})] - \int_P f \right| \\ &\leq \epsilon + \sum \frac{|\gamma(t_j) - \gamma(t_{j-1})|}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} |f \circ P(t) - f \circ \gamma(t_j)| dt \end{aligned}$$

From (4), we get

$$\begin{aligned} \left| \int_P f - \int_P f \right| &< \epsilon + \epsilon \sum |\gamma(t_j) - \gamma(t_{j-1})| \\ &= \epsilon (1 + V(\gamma)) \end{aligned}$$

Case II  $\gamma$  is arbitrary open set:

Since  $\gamma$  is cpl,  $\exists 0 < \gamma < \delta$  s.t.  $(\gamma, \delta)$ .

Choose  $\delta > 0$  s.t.  $|\gamma(s) - \gamma(t)| < \gamma$ ,  $|s - t| < \delta$ .

If  $P = \{t_0 < t_1 < \dots < t_n\}$  is a partition of  $[a, b]$  with  $\|P\| < \delta$ , then

$$|\gamma(t) - \gamma(t_{j-1})| < \gamma \quad t_{j-1} \leq t \leq t_j$$

That is,  $\gamma_j : [t_{j-1}, t_j] \rightarrow G$   $\gamma_j(t) = \gamma(t)$ , (iii)

then  $\gamma_j \in B(\gamma(t_{j-1}), r)$  for  $1 \leq j \leq n$ .

By case I, there is a polygon  $P_n$  st.

$P_{j-1} : [t_{j-1}, t_j] \rightarrow B(\gamma(t_{j-1}), r)$  with

$$P_{j-1}(t_{j-1}) = \gamma(t_{j-1}), \quad P_j(t_j) = \gamma(t_j)$$

$$\& \left| \int_{\gamma_j} f - \int_{P_j} f \right| < \epsilon/n.$$

If  $P(t) = P_j(t)$  on  $[t_{j-1}, t_j]$ , then

$$\left| \int_{\gamma} f - \int_P f \right| < \epsilon.$$



Power series rep<sup>n</sup> of analytic function

(112)

Let  $\varphi: [a, b] \times [c, d] \rightarrow \mathbb{C}$  be cont &

define  $g: [c, d] \rightarrow \mathbb{C}$  by

$$g(t) = \int_a^b \varphi(s, t) ds \quad \text{--- (1)}$$

Then  $g$  is cont. If  $\frac{\partial \varphi}{\partial t}$  exists & cont, then  $g'$  exists & cont.

Proof:  $g(t+h) - g(t) = \int_a^b (\varphi(s, t+h) - \varphi(s, t)) ds$

For  $\epsilon > 0$ ,  $\exists \delta > 0$  s.t.

$$|h| < \delta \Rightarrow |\varphi(s, t+h) - \varphi(s, t)| < \epsilon / (b-a)$$

$$\therefore |g(t+h) - g(t)| \leq \int_a^b \frac{\epsilon}{b-a} ds = \epsilon \quad \text{if } |h| < \delta.$$

$\Rightarrow g$  is cont on  $[c, d]$ .

If  $\frac{\partial \varphi}{\partial t}$  is cont, then

$$\left| \frac{g(t+h) - g(t)}{h} - \int_a^b \frac{\partial \varphi}{\partial t}(s, t) ds \right| =$$

$$= \int_a^b \left[ \frac{\varphi(s, t+h) - \varphi(s, t)}{h} - \frac{\partial \varphi(s, t)}{\partial t} \right] ds \quad (113)$$

now, we can assume by going through real and imaginary parts, that  $\varphi$  is real valued.

$$= \int_a^b \left[ \frac{\partial \varphi}{\partial t}(s, t+h) - \frac{\partial \varphi}{\partial t}(s, t) \right] ds$$

Since  $\frac{\partial \varphi}{\partial t} = \varphi_2$  is unif. cont. for  $t > 0$ ,  
 $\exists \delta > 0$  s.t.

$$|h| < \delta \Rightarrow |\varphi_2(s, t+h) - \varphi_2(s, t)| < \epsilon$$

$$\Rightarrow \left| \frac{\varphi(t+h) - \varphi(t)}{h} - \int_a^b \varphi_2(s, t) ds \right| < \epsilon (b-a)$$

$$\Rightarrow g'(t) = \int_a^b \frac{\partial \varphi}{\partial t}(s, t) ds.$$

Since  $\frac{\partial \varphi}{\partial t}$  is cont,  $g'(t)$  is also cont.

Ex. Show that  $\int_0^{2\pi} \frac{e^{iz}}{e^z - z} dz = 2\pi$  if  $|z| < 1$ .

$$\text{let } \varphi(s, t) = \frac{e^{iz}}{e^s - z}; \quad 0 \leq t \leq 1, \quad 0 \leq s \leq 2\pi$$

Then  $\varphi$  is continuously differentiable on  $|z| < 1$ .

$$g(t) = \int_0^{2\pi} \varphi(se^{it}) ds, \text{ then } g(0) = g(1) = 2\pi \quad (114)$$

We show that  $g$  is constant, and  $g(1) = g(0) = 2\pi$ .

$$\text{Now, } g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - z)^2} ds$$

For fixed  $t$ , let  $\bar{\varphi}(s) = \frac{ze^{is}}{e^{is} - z}$ . Then

$$\bar{\varphi}'(s) = \frac{-zi(e^{is})}{(e^{is} - z)^2} = \frac{ze^{is}}{(e^{is} - z)^2}$$

$$g'(t) = \bar{\varphi}(2\pi) - \bar{\varphi}(0) = 0 \Rightarrow g \text{ is constant.}$$

Prop: Let  $f: G \rightarrow \mathbb{C}$  be analytic &

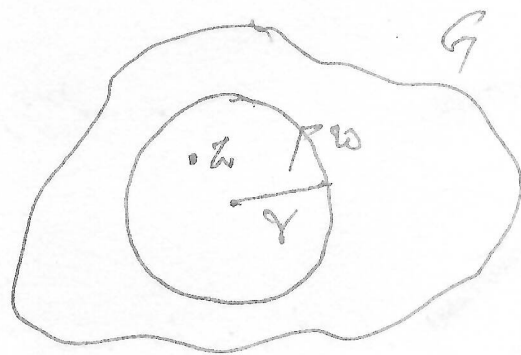
$\bar{B}(a, r) \subset G$ , if  $\gamma(t) = a + re^{it}$ ,  $0 \leq t \leq 2\pi$ ,

then  $f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw$  for  $|z - a| < r$ .

Proof: By Cauchy's Integral Formula

$$\tilde{f}_1 = \left\{ \frac{1}{r} (z - a) : z \in G \right\}$$

$$f(z) = f(a + rz)$$



Wlog. we can assume that  $a=0$ ,  $r=1$ .

That is,  $\overline{B(0,1)} \subset G$ .

(1.5)

Fix  $z$ ,  $|z| < 1$ , we show that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta}) e^{i\theta}}{e^{i\theta} - z} d\theta \end{aligned}$$

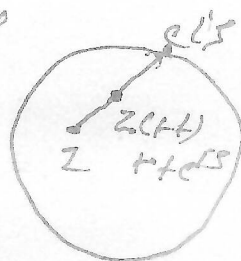
$$\text{we } 0 = \int_0^{2\pi} \int \left[ \frac{f(e^{i\theta}) e^{i\theta}}{e^{i\theta} - z} - f(z) \right] d\theta.$$

$$\text{Let } \varphi(s,t) = \frac{f(z + t(e^{i\theta} - z)) e^{i\theta}}{e^{i\theta} - z} - f(z)$$

$$0 \leq t \leq 1, \quad 0 \leq \theta \leq 2\pi$$

$$\text{Since } |z + t(e^{i\theta} - z)| = |z(1-t) + te^{i\theta}| < 1$$

(the convex combination of two  
pts in  $G$  is in  $G$ )



Then  $\varphi$  is well defined and continuous.

$$\text{Let } g(t) = \int_0^{2\pi} \varphi(s,t) d\theta. \text{ Then}$$



$$g(0) = \int_0^{2\pi} \varphi(s, 0) ds = \int_0^{2\pi} \int \frac{f(z) e^{is}}{e^{is} - z} f(\bar{z}) ds$$

We claim  $g$  is constant

(18)

By Leibniz's rule,

$$g'(t) = \int_0^{2\pi} \varphi_2(s, t) ds, \text{ where}$$

$$\varphi_2(s, t) = e^{is} f'(z + t(e^{is} - z)).$$

Also,  $\Phi'(s) = \frac{-it(z + t(e^{is} - z))}{+}$  is  
primitive of  $\varphi_2(s, t)$ . That

$$\Phi'(s) = \varphi_2(s, t)$$

$$\text{So } g'(t) = \Phi(2\pi) - \Phi(0) = 0$$

$\Rightarrow g$  is constant

$$g(0) = g(1) = 0 \Rightarrow 2\pi f(z) = \int_0^{2\pi} \frac{f(e^{is}) e^{is}}{e^{is} - z} ds.$$

Lemma Let  $\gamma$  be a rectifiable curve in  $\mathbb{C}$   
and suppose  $F_n, F$  are conts on  $\gamma$   
if  $F_n \rightarrow F$  unif., then  $\int_{\gamma} F = \lim_{n \rightarrow \infty} \int_{\gamma} F_n$ .

Proof! By uniform conv, for  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$

$$\text{s.t. } |F_n(w) - F(w)| < \epsilon / V(\gamma) \quad \forall w \in \gamma \\ \forall n \geq N.$$

$$\begin{aligned} \left| \int_{\gamma} F_n - \int_{\gamma} F \right| &= \left| \int_{\gamma} (F_n - F) \right| & (117) \\ &\leq \int_{\gamma} |F_n(w) - F(w)| |dw| \\ &< (\epsilon / V(\gamma)) \cdot \int_{\gamma} |dw| = \frac{\epsilon}{V(\gamma)} \cdot V(\gamma) = \epsilon \end{aligned}$$

Thm: Let  $f$  be analytic in  $B(a, R)$ , then

$$f(z) = \sum a_n (z-a)^n \quad \text{for } |z-a| < R,$$

where  $a_n = \frac{1}{n!} f^{(n)}(a)$  & the series has radius of conv  $\geq R$ .

Proof: let  $0 < r < R$ , so that  $\overline{B(a, r)} \subset B(a, R)$

$$\text{let } \gamma(t) = a + re^{it}, \quad 0 \leq t \leq 2\pi.$$

$$\text{then } f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw, \quad \text{for } |z-a| < r.$$

note that  $|z-a| < r$ , so if on  $|w-a| = r$

Note that  $|z-a| < r$ ,  $w \in \gamma$

$$\frac{1}{w-z} = \frac{1}{w-a} \frac{1}{1 - \frac{z-a}{w-a}} = \frac{1}{w-a} \sum_{n=0}^{\infty} \left( \frac{z-a}{w-a} \right)^n \quad \text{--- (6)}$$

$$\text{So } \frac{f(w)}{w-z} = \sum \frac{f(w)}{(w-a)^{n+1}} (z-a)^n \quad \text{--- (7)}$$

(18)

Also,  $|z-a| < r$ ,  $w \in \gamma$

$$\frac{|f(w)| |z-a|^n}{|w-a|^{n+1}} \leq \frac{M}{r} \left( \frac{r}{r} \right)^n$$

where  $M = \sup \{ |f(w)| : |w-a| = r \}$

Since  $\frac{|z-a|}{r} < 1$ , by Weierstrass M-test

the series  $\sum \frac{f(w) (z-a)^n}{(w-a)^{n+1}}$  is Conv.

$$\text{and } f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dz = \sum \frac{1}{2\pi i} \left( \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dz \right) (z-a)^n$$

$|z-a| < r$

If we set  $a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dz$ , and

we know  $a_n = \frac{1}{n!} f^{(n)}(a)$ , we get  $f(z) = \sum \frac{n!}{n!} \frac{f^{(n)}(a)}{n!} (z-a)^n$

Thus, the power series  $f(z) = \sum a_n(z-a)^n$   
 converges at  $z$ ,  $|z-a| < r < R$ , it (119)  
 follows that s.c. of the series is  $\geq R$ .

Cor: If  $f: G \rightarrow \mathbb{C}$  is analytic,  $a \in G$   
 then  $f(z) = \sum a_n(z-a)^n$ , for  $|z-a| < R$ ,  
 where  $R = d(a, \partial G)$ .

Proof: Let  $R = d(a, \partial G)$ ,  $G$ -open, then  
 $B(a, R) \subset G$

So  $f$  is analytic on  $B(a, R)$ , and the  
 result follows from the theorem.

\* Since power series is only differentiable,  
 it follows that every analytic function is  
 only differentiable.

Cor: If  $f: G \rightarrow \mathbb{C}$  analytic &  $B(a, r) \subset G$   
 then 
$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw$$
  
 where  $\gamma = a + re^{it}$ ,  $0 \leq t \leq 2\pi$ .



Cauchy estimate:

Let  $f$  be analytic in  $B(a, R)$  and

$$|f(z)| \leq M \quad \forall z \in B(a, R).$$

(120)

$$\text{Then } |f^{(n)}(a)| \leq \frac{n! M}{R^n}$$

Proof: For  $r < R$ ,

$$\begin{aligned} |f^{(n)}(a)| &= \left| \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} dw \right| \\ &\leq \left| \frac{n!}{2\pi i} \right| \int_0^{2\pi} \frac{M}{r^{n+1}} \cdot r dt = \frac{n! M}{r^n} \rightarrow \frac{n! M}{R^n} \end{aligned}$$

Prop: Let  $f$  be analytic in  $B(a, R)$ , and suppose that  $\gamma$  is a closed curve (rectifiable) in  $B(a, R)$ . Then  $\int_{\gamma} f = 0$ .

Proof: We show that  $f$  has primitive

$$f(w) = \sum a_n (z-a)^n, \quad |z-a| < R.$$

$$\begin{aligned} \text{Let } F(z) &= \sum_{n=0}^{\infty} \left( \frac{a_n}{n+1} \right) (z-a)^{n+1} \\ &= (z-a) \sum_{n=0}^{\infty} \left( \frac{a_n}{n+1} \right) (z-a)^n \end{aligned}$$

The Cauchy integral has same  
value of  $\log z$ , since  $(1+z)^{1/n} = 1$ .

Assume  $F$  is defined on  $B(a, R)$  & (12)

$$F'(z) = f(z) \quad \text{on } |z-a| < R$$

$$\int_{\gamma} f(z) dz = \int_a^b F'(\gamma(t)) \gamma'(t) dt$$

$$= \int_a^b \frac{d}{dt} (F \circ \gamma) dt$$

$$= F(\gamma(b)) - F(\gamma(a)) = 0.$$

ML-inequality:

Let  $\gamma: [a, b] \rightarrow G$  be a rectifiable curve  
&  $f: G \rightarrow \mathbb{C}$  be cont. Then

$$\left| \int_{\gamma} f(z) dz \right| \leq M \int_{\gamma} |dz| = M V(\gamma) = ML,$$

where  $L = \int_a^b |\gamma'(t)| dt$ , &  $\gamma$  smooth, where

$$M = \sup_{z \in \gamma} |f(z)| < \infty$$

Ex.  $|f(t)| = 2e^{it}$ ,  $t \in [0, \pi/2]$

$f(z) = \frac{z+4}{z^3-1}$ , then by ML-inequality

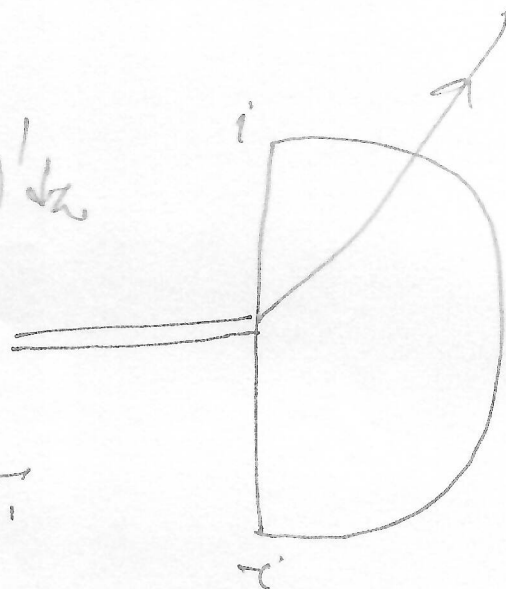
$$\left| \int_{\gamma} f(z) dz \right| \leq \frac{6\pi}{7}$$

(22)

Ex.  $\int_{\gamma} \frac{1}{z} dz = \int_{-i}^i (\text{Log}(z))' dz$

$$= \text{Log}(i) - \text{Log}(-i)$$

$$= i\pi/2 - (-i\pi/2) = i\pi$$



because  $\frac{1}{z} = (\text{Log } z)'$  on  $z \in \mathbb{C} \setminus [0, \infty)$ .

Ex.  $\int_{\gamma} \frac{dz}{z^n} = -(n-1) \left\{ \frac{1}{z_2^{n-1}} - \frac{1}{z_1^{n-1}} \right\}$ ,  $\neq n=1$

Ex.  $\int_{\gamma} z^2 dz = \frac{z_2^3 - z_1^3}{3}$  ( $\because F(z) = \frac{z^3}{3}$   
 $f(z) = F'(z)$ )

Ex.  $\int_{-i\pi}^{i\pi} \log z dz = 2\pi i \ln 1 \pi.$

## Zeros of analytic function

(123)

Let  $p$  &  $q$  be two polys, then  $\exists$  polys  $s$  &  $r$  st

$$p(z) = s(z)q(z) + r(z)$$

where  $\deg r(z) < \deg q(z)$ .

In particular, if  $p(a) = 0$ , then choose  $q(z) = z - a$ , then

$$p(z) = (z - a)s(z) + r(z),$$

where  $r(z)$  must be a const. But

$$0 = p(a) = r(a) \Rightarrow r = 0.$$

In this way, we get

$$p(z) = (z - a)^m f(z), \text{ where}$$

$$1 \leq m < \deg p(z) \text{ & } f(a) \neq 0.$$

$$\text{& } \deg f(z) = \deg p(z) - m.$$

def<sup>n</sup>: If  $f: G \rightarrow \mathbb{C}$  is analytic & a ch,  
be such that  $f(a) = 0$ , then we say  
a is of multiplicity  $m$  if  $\exists g(z)$  st



$$f(z) = (z-a)^m g(z), \quad g(a) \neq 0. \quad (124)$$

ex. If  $p(z)$  is a poly of degree  $n$  &  $a_1, \dots, a_k$  are distinct zeros of  $p(z)$ , then

$p(z) = (z-a_1)^{m_1} \dots (z-a_k)^{m_k} S(z)$ , where  $S(z)$  has no zero (if will be followed from FTA that  $S(z)$  is constant.)

Recall that for an entire function,

$$\text{then } f(z) = \sum a_n z^n \quad \text{---} \quad (*)$$

with radius of conv. infinity.

In light of (\*), an entire function can be considered as a poly of infinite degree.

So the question is that can theory of ~~ent~~ poly be generalized to entire functions? For example, can an entire function be factored? we discuss it later.

Another property of poly is, no non-constant poly is bounded. Indeed,

(125)

$$p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$$

$$\lim_{z \rightarrow \infty} p(z) = \lim_{z \rightarrow \infty} z^n \left[ 1 + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right]$$

$$= \infty \text{ (finite)} = \infty$$

The fact also holds for entire function.

Liouville's theorem:

If  $f$  is bounded entire, then  $f$  is constant.

Proof: By Cauchy estimate

$$|f(z)| \leq \frac{M}{R} \rightarrow 0 \text{ as } R \rightarrow \infty$$

$$\text{where } M = \sup_{z \in \mathbb{C}} |f(z)| < \infty.$$

$$\Rightarrow f(z) = 0 \Rightarrow f \text{ is constant.}$$

That is, non-constant entire functions are not bounded.

## Fundamental theorem of Algebra (FTA)

If  $p(z)$  is a non-constant poly, then  
 $\exists a \in \mathbb{C}$  st  $p(a) = 0$ . (126)

Proof: Suppose  $p(z) \neq 0, \forall z \in \mathbb{C}$ , then

$f(z) = \frac{1}{p(z)}$  is an entire function

$$\text{and } \lim_{z \rightarrow \infty} f(z) = \lim_{z \rightarrow \infty} \frac{1}{p(z)} = 0$$

$\Rightarrow f$  is bounded & hence by  
Liouville's theorem,  $f$  is constant  
 $\Rightarrow p$  is constant, a contradiction.

Ex. Let  $f$  be analytic function on domain  
(open & conn)  $G$  and  $\exists a \in G$  st

$|f(a)| < |f(z)|, \forall z \in G$ , then show  
that either  $f(a) = 0$  or  $f$  is constant.

If  $f(a) \neq 0$ , then  $\left| \frac{f(a)}{f(z)} \right| < 1 \Rightarrow g(z) = \frac{f(a)}{f(z)}$   
is bounded on  $G$ , etc.

Any poly  $p(z)$  will satisfy

(127)

$$p(z) = c(z-a_1)^{K_1} \cdots (z-a_m)^{K_m}.$$

That is, every poly over complex field can be completely factorized linearly. Thus, every complex poly has exactly  $n$  roots.

Note that if  $p$  is a poly  $f, a \in \mathbb{C}$ , then  $\exists z \in \mathbb{C}$  st.  $p(z) = a$ . This follows by FTA.

Consider  $g(z) = p(z) - a$ , then  $\exists z_0 \in \mathbb{C}$  st.  $g(z_0) = 0 \Rightarrow p(z_0) = a$ .

However, this is not the case of exp. function  $e^z$ , which never zero.

But we see that this is the worst that can happen. We show that an entire function can omit at most one point

i.e.  $\text{Range}(f) = \mathbb{C} \setminus \{z_0\}$  for some  $z_0 \in \mathbb{C}$ .  
(Known as Picard Little Theorem)



(Known as Liouville's theorem)

Ex. An entire function cannot be bounded.

If  $f: \mathbb{C} \rightarrow \mathbb{C}$  is entire & non-constant, then  
 $\nexists \epsilon > 0$  st  $|f(z)| \geq \epsilon \quad \forall z \in \mathbb{C}$  (128)

If  $|f(z)| \geq \epsilon, \forall z \in \mathbb{C} \Rightarrow \left| \frac{\epsilon}{f(z)} \right| \leq 1,$

As  $\frac{\epsilon}{f(z)}$  is entire & bdd  $\Rightarrow \frac{\epsilon}{f(z)}$  is constant

We should not make an analogy  
between functions defined on an open  
set  $G$  and a poly which is defined on  $\mathbb{C}$ ,  
rather restrict poly. to  $G$ .

Consider  $f(z) = \cos\left(\frac{1+z}{1-z}\right) : |z| < 1$

Note that  $\frac{1+z}{1-z}$  maps  $D = \{z : |z| < 1\}$  onto

$G = \{z : \operatorname{Re} z > 0\}$ .

The zeros of  $f$  are the points  $\frac{j\pi - z}{\pi + z} : n \in \mathbb{Z}$

So  $f$  has infinitely many zeros which approach  
to 1 which is not in the domain of  $f$

analyticity of  $f$ . We have the following result known as Identity Theorem in this regard. (129)

Thm: Let  $G$  be an open conn. set in  $\mathbb{C}$  and let  $f: G \rightarrow \mathbb{C}$  be analytic.

Then F.A.E:

- (i)  $f \equiv 0$
- (ii)  $\exists$  a point  $a \in G$  s.t.  $f^n(a) = 0, \forall n \geq 0$
- (iii)  $\{z \in G : f(z) = 0\}$  has a limit point.

Proof: Clearly, (i)  $\Rightarrow$  (ii) & (iii).

(iii)  $\Rightarrow$  (ii): let  $a \in G$  be a limit point of  $Z = \{z \in G : f(z) = 0\}$ .

Let  $R > 0$  be s.t.  $B(a, R) \subset G$

Since  $a$  is a limit point of  $Z$  &  $f$  is const, it follows that  $f(a) = 0$ .

$\therefore \exists a_n \in Z$  s.t.  $a_n \rightarrow a \Rightarrow f(a) = \lim f(a_n) = 0$

Suppose  $\exists n \in \mathbb{N}$  s.t.  $f(a) = f'(a) = \dots = f^{(n)}(a) = 0$

but  $f^n(a) \neq 0$ . Then

$$f(z) = \sum_{k=n}^{\infty} a_k (z-a)^k, \quad |z-a| < R$$

(130)

✓  $f(z) = (z-a)^n g(z)$ , where  $g$  is analytic

$$\& f^n(a) = g(a) \neq 0.$$

Since  $g$  is cont,  $\exists \delta > 0$  st

$$g(z) \neq 0 \text{ for } |z-a| < \delta$$

But  $a$  is limit point of  $Z$

$\exists b (\neq a) \in B(a, \delta)$  st

$$f(b) = 0$$

$$0 = (b-a)^n g(a)$$

$$\Rightarrow g(a) = 0 \text{ a contradiction.}$$

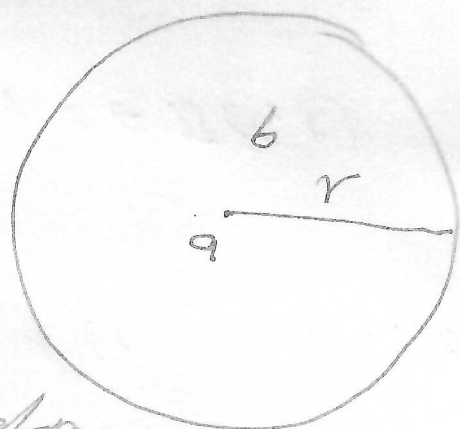
Hence  $\nexists$  such  $n$  with  $f^n(a) \neq 0$ .

(ii)  $\Rightarrow$  (i):

$$\text{let } A = \{z \in G : f^n(z) = 0, \forall n \in \mathbb{N}\}.$$

Then  $a \in A \Rightarrow A \neq \emptyset$ .

We claim  $A$  is both closed & open.



Since  $f^n$  are continuous, for  $z \in \bar{A}$ ,  
 $\leftarrow z_k \rightarrow z, f^n(z_k) = 0 \Rightarrow f^n(z) = 0.$  (13)  
 Hence  $A$  is closed.

To show  $A$  is open, let  $c \in A$ , then  
 $\exists R > 0$  st  $B(c, R) \subset G$ , and

$$f(z) = \sum a_n (z-c)^n, |z-c| < R$$

But  $c \in A \Rightarrow a_n = \frac{1}{n!} f^n(c) = 0, \forall n \geq 0$   
 $\Rightarrow f(z) = 0 \quad \forall z \in B(c, R).$

$$\Rightarrow B(c, R) \subset A$$

$$\Rightarrow B(c, R) \subset A$$

$\Rightarrow A$  is open.

Hence,  $A = G \Rightarrow f \equiv 0.$

Ex: If  $f, g$  are analytic on  $G$  then  
 $f \equiv g$  iff  $\{z \in G: f(z) = g(z)\}$  has limit points.

Ex:  $f$  is analytic on  $G$ ,  $f \neq 0$ , with  
 $f(a) = 0$ , then  $\exists n \geq 1$  st



$$f(z) = (z-a)^n g(z), \quad g(a) \neq 0$$

(132)

and  $g$  is analytic.

That is,  $\exists B(a, R)$  st  $f(z) \neq 0, \forall z \in B(a, R) \setminus \{a\}$ .

$\Rightarrow$  zeros of  $f(z)$  are isolated.

There is one instance when analogy between poly. and analytic function reverse, which is not so transparent for polynomial

Maximum Modulus Thm:

Let  $G$  be open and conn. &  $f: G \rightarrow \mathbb{C}$  be analytic. If  $\exists a \in G$  st  $|f(a)| \geq |f(z)| \forall z \in G$ , then  $f$  is constant.

Proof: let  $\bar{B}(a, r) \subset G$ ,  $0 < r < \infty$ , and  $\gamma(t) = a + re^{it}$ . Then by Cauchy formula;

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} dw = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt$$

$$\Rightarrow |f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{it})| dt \leq |f(a)|$$

$$\Rightarrow \int_0^{2\pi} [|f(a)| - |f(a + re^{it})|] dt = 0.$$

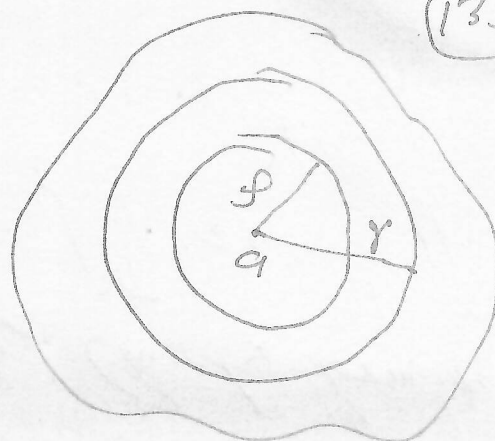
$$\Rightarrow |f(z)| = |f(a + \rho e^{it})| \quad \forall t \in [0, 2\pi]$$

and  $\forall \rho < r$ .

(133)

$\Rightarrow |f|$  is constant on  $B(a, r)$

$\Rightarrow f$  is const. on  $B(a, r)$ .



By identity thm,  $f$  is const. on  $G$ .

Cor: If  $f$  is cont. on  $\partial G$ , and  $G$  is a bounded open conn. set, then  $\exists z_0 \in \partial G$  s.t.

$$\sup_{z \in G} |f(z)| = |f(z_0)|.$$

Pr: Since  $f$  is cont. on  $G \cup \partial G$  (compact),  $f$  will attain its sup. on  $G \cup \partial G$ . By

f.l.  $\exists z_0 \in G \cup \partial G$  s.t.

$$|f(z_0)| = \sup_{z \in G} |f(z)|, \quad G = G \cup \partial G$$

By MMT,  $z_0 \notin G \Rightarrow z_0 \in \partial G$ .

Cor: (Maximum Modulus thm)

Let  $G$  be an open & conn. set in  $\mathbb{C}$ .



ans  $f: G \rightarrow \mathbb{C}$  be analytic. Let

$$a \in G \text{ st } 0 < |f(a)| \leq |f(z)|, \forall z \in G.$$

(134)

Then  $f$  is constant.

Pf:  $g = \frac{1}{f}$ , then  $|g(z)| \leq |g(a)|, \forall z \in G$ .

$\Rightarrow g$  is const  $\Rightarrow f$  is const.

Remark: If  $f: \mathbb{C} \rightarrow \mathbb{C}$  be entire, then  $f(\mathbb{C})$  cannot skip an open set in  $\mathbb{C}$ , then  $f$  is constant.

Ex. If  $u$  is a harmonic function on  $\mathbb{C}$ ,  
s.t.  $u(z) \geq 0 \forall z \in \mathbb{C}$ , then  $u$  is const.

Since  $u$  is a harmonic function on  $\mathbb{C}$ ,  
it must have a harmonic conjugate  
by  $v$  st  $f = u + iv$  is analytic.

$$|e^f| = e^u \geq 1 \Rightarrow |e^f| \leq 1$$

By Liouville's Thm,  $e^f$  is const

$\Rightarrow u$  is const.

Ex. let  $f, g$  be analytic on domain  $D$  st.

$$f(z)g(z) = 0 \quad \forall z \in D.$$

(135)

Show that either  $f \equiv 0$  or  $g \equiv 0$ .

St.  $\exists$  two points  $z_1, z_2$  st.  $f(z_1) = 0, g(z_1) \neq 0$   
or  $f(z_2) \neq 0, g(z_2) = 0$ . Then

$\exists R > 0$  st.  $g(z) \neq 0, \forall z \in B(z_1, R)$ .

$\Rightarrow f(z) = 0, \forall z \in B(z_1, R)$

By identity, thm,  $f \equiv 0$  which is  
contradict that  $f(z_2) \neq 0$ .

Thus,



## Index of closed curve:

136

If  $\gamma(t) = a + re^{2\pi i n t}$ ,  $0 \leq t \leq 1$ , then

$$\int_{\gamma} \frac{dz}{z-a} = \int_0^1 \frac{r e^{2\pi i n t} 2\pi i n dt}{r e^{2\pi i n t}} = 2\pi i n$$

$$\text{i.e. } \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} = n \text{ (integer)}$$

Prop: If  $\gamma: [0,1] \rightarrow \mathbb{C}$  is closed smooth rectifiable curve &  $a \notin \gamma$ , then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} \text{ is an integer.}$$

pf: Let  $g: [0,1] \rightarrow \mathbb{C}$  be defined by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)-a} ds.$$

$$\text{Then } g(0) = 0 \text{ \& } g(1) = \int_{\gamma} \frac{dz}{z-a}.$$

$$\text{we have } g'(t) = \frac{\gamma'(t)}{\gamma(t)-a} \quad 0 \leq t \leq 1.$$

Notice that

$$\begin{aligned} \frac{d}{dt} (e^{-g}(\gamma-a)) &= e^{-g} (-g')(\gamma-a) + e^{-g} \cdot \gamma' \\ &= e^{-g} (-\gamma' + \gamma') = 0. \end{aligned}$$

$$\Rightarrow e^{-g}(\gamma-a) \text{ is constant.}$$

$$\Rightarrow e^{-g(0)}(\gamma(0)-a) = (\gamma(1)-a)e^{-g(1)}$$

(137)

$$\text{But } \gamma(0) = \gamma(1) \Rightarrow e^{-g(1)} = 1$$

$$\Rightarrow g(1) = 2\pi i n$$

$$\text{Hence } \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} = n.$$

Def<sup>n</sup>: If  $\gamma$  is a closed rectifiable curve in  $\mathbb{C}$ ,

and  $a \notin \gamma$ , then

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

is known as index of  $\gamma$  about  $a$ .

(i) If  $\gamma: [0,1] \rightarrow \mathbb{C}$  is a curve then

$$(-\gamma)(t) = \gamma(1-t)$$

(ii) If  $\gamma, \sigma$  are two curves on  $[0,1]$  s.t.

$$\gamma(1) = \sigma(0), \text{ then } \gamma + \sigma \text{ is a curve}$$

given by

$$(\gamma + \sigma)(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ \sigma(1-2t) & 1/2 \leq t \leq 1. \end{cases}$$

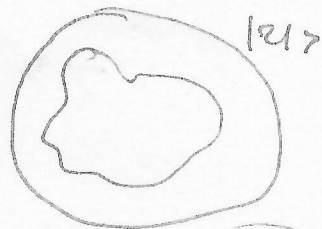
$$(iii) n(\gamma, a) = -n(-\gamma, a)$$

$$(iv) n(\gamma + \sigma, a) = n(\gamma, a) + n(\sigma, a), \forall a \notin \gamma \cup \sigma.$$

The number  $n(\gamma, a)$  is also known as winding no.



Ex: Let  $\gamma$  be a closed rectifiable curve in  $\mathbb{C}$   
and set  $G = \mathbb{C} \setminus \gamma$ . Since  $\gamma$  is cpt,  
 $\exists R > 0$  st  $\{z: |z| > R\} \subset G$ .



Thm: Let  $\gamma$  be a closed rectifiable  
curve in  $\mathbb{C}$ . Then  $n(\gamma, a)$  is constant  
for  $a$  belongs to a component ~~of~~ of  
 $G = \mathbb{C} \setminus \gamma$ . Also  $n(\gamma, a) = 0$  for  $a$   
belongs to the unbounded component of  $G$ . (138)

Proof: Define  $f: G \rightarrow \mathbb{C}$  by  
$$f(a) = n(\gamma, a).$$

We claim that  $f$  is continuous. If so,  
then  $f(D)$  is connected for each component  
 $D$  of  $G$ .

Since  $f(D) \subset \mathbb{Z}$

$\Rightarrow f(D) = \{n_0\} \Rightarrow f$  is const. on.

Note that the components of  $G$  are open.

Fix  $a \in G$ , let  $\gamma = d(a, \gamma)$ . Then

(139)

for  $|a-b| < \delta < \frac{1}{2}\gamma$ , we have

$$|f(a) - f(b)| = \frac{|a-b|}{2\pi} \int_{\gamma} \frac{|dz|}{|z-a||z-b|}$$

Note that  $|z-a| > \gamma > \frac{1}{2}\gamma$  &  $|z-b| > \frac{1}{2}\gamma$

$$\Rightarrow |f(a) - f(b)| < \frac{2\delta}{\pi\gamma^2} V(\gamma) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

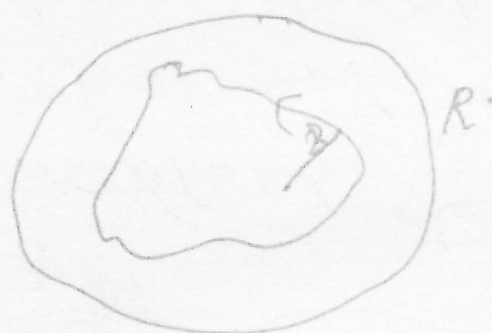
Let  $V$  be an unbounded component.

Then  $\exists R > 0$  s.t.  $\cup \{z: |z| > R\}$

choose a pt.  $a$  with

$|a| > R$ , and

$$|z-a| > \frac{2V(\gamma)}{\epsilon}$$



uniformly for  $z \in \gamma$ .

Then  $n(\gamma, a) \leq \int_{\gamma} \frac{\epsilon}{V(z)} |dz| < \epsilon/2 < \epsilon$

That is,  $n(\gamma, a) < \epsilon$  if  $|z-a| > \frac{2V(\gamma)}{\epsilon}$



$$\Rightarrow n(\gamma, a) \rightarrow 0 \text{ as } a \rightarrow \infty$$

140

Since  $n(\gamma, a)$  is const on  $U^*$

$$\Rightarrow n(\gamma, a) = 0.$$

Lemma: Let  $\gamma$  be a closed rectifiable curve and suppose  $\varphi: \gamma \xrightarrow{\text{cont}} \mathbb{C}$ . For  $m \in \mathbb{Z}$ ,

$$\text{let } F_m(z) = \int_{\gamma} \frac{\varphi(w)}{(w-z)^{m+1}} dz, \quad z \notin \gamma.$$

Then each  $F_m$  is analytic on  $\mathbb{C} \setminus \gamma$

$$\text{and } F_m'(z) = m F_{m+1}(z).$$

Proof: As similar to the previous thm, it can be shown that each  $F_m$  is cont, since  $\gamma$  is compact,  $\varphi$  is bdd on  $\gamma$ .

$$\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} = (z-a) \sum_{k=0}^{m-1} \frac{1}{(w-z)^{m-k} (z-a)^k} \quad (*)$$

Let  $a \in G = \mathbb{C} \setminus \gamma$  &  $z \in G, z \neq a$

Then from (\*), we have

$$\frac{F_m(z) - F_m(a)}{z-a} = \int_{\gamma} \frac{\varphi(w)(w-a)^{-1}}{(w-z)^m} dw + \dots + \int_{\gamma} \frac{\varphi(w)(w-a)^{-m}}{(w-z)} dw$$

Since  $a \notin \gamma$ ,  $\varphi(w)(w-a)^{-k}$  is cont. on  $\gamma$ . (14)

Hence, each integral in the RHS defines a cont. function & letting  $z \rightarrow a$ , we get

$$F_m'(a) = \int_{\gamma} \frac{\varphi(w)}{(w-a)^{m+1}} dw + \dots + \int_{\gamma} \frac{\varphi(w)}{(w-a)^{m+1}} dw \\ = m F_{m+1}(a).$$

Cauchy integral formula:

Let  $f$  be analytic on an open set  $G$ . If  $\gamma$  is a closed rectifiable curve in  $G$

st-  $n(\gamma, z) = 0$ ,  $\forall z \in \mathbb{C} \setminus G$ , then for  $a \in G \setminus \gamma$ , we have

$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int \frac{f(z)}{z-a} dz.$$



Proof: Define  $\varphi: G \times G \rightarrow \mathbb{C}$  by

(142)

$$\varphi(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(z) & \text{if } z = w \end{cases}$$

Then each  $\varphi$  is const, and for each  $w \in G$ ,  $z \mapsto \varphi(z, w)$  is analytic.

Let  $H = \{w \in G : n(\gamma, w) = 0\}$ .

Since  $n(\gamma, w)$  is continuous integer-valued function,  $H$  is open.

Also,  $H \cup G = \mathbb{C}$

Define  $g: \mathbb{C} \rightarrow \mathbb{C}$  by

$$g(z) = \begin{cases} \int_{\gamma} \varphi(z, w) dw & \text{if } z \in G \\ \int_{\gamma} \frac{f(w)}{w - z} dw & \text{if } z \in H \end{cases}$$

If  $z \in G \cap H$ , then

$$\int_{\gamma} \varphi(z, w) dw = \int_{\gamma} \frac{f(w) - f(z)}{z - w} dw$$

$$\begin{aligned}
 &= \int_{\gamma} \frac{f(w)}{w-z} dw - f(z) n(\gamma, z) 2\pi i \\
 &= \int_{\gamma} \frac{f(w)}{w-z} dw - f(z) \cdot 0 \cdot 2\pi i \quad (143)
 \end{aligned}$$

Hence,  $g$  is well defined.

By previous lemma,  $g$  is analytic on  $D$ .

That is  $g$  is an entire function.

By previous thm, it contains a subseq. as in Cas.

Since  $f$  is bounded &  $\lim_{z \rightarrow \infty} (w-z) = 0$  uniformly for  $w \in \gamma$ , it follows that

$$\lim_{z \rightarrow \infty} g(z) = \lim_{z \rightarrow \infty} \int_{\gamma} \frac{f(w)}{w-z} dw = 0$$

By Liouville's thm,  $g$  is constant. If  $a \in G \setminus \gamma$ , then

$$0 = \int \frac{f(z) - f(a)}{z-a} dz$$



$$= \int \frac{f(z)}{z-a} dz = f(a) n(\gamma, a) 2\pi i$$

144

$$\Rightarrow f(a) n(\gamma, a) = \frac{1}{2\pi i} \int \frac{f(z)}{z-a} dz.$$

Cor: Let  $f$  be analytic on an open set  $G$ . If  $\gamma_1, \dots, \gamma_m$  are closed rectifiable curves in  $G$  s.t.

$$n(\gamma_1, w) + \dots + n(\gamma_m, w) = 0$$

$\forall z \in \mathbb{C} \setminus G$ , then for  $a \in G \setminus \{\gamma\}$

$$f(a) \sum_{k=1}^m n(\gamma_k, a) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z-a} dz$$

Proof: The proof is similar to theorem.

Cauchy's theorem:

Let  $f$  be analytic on an open set  $G$ .

If  $\gamma$  is a closed rectifiable curve s.t.

$n(\gamma, w) = 0, \forall w \in \mathbb{C} \setminus G$ , then  $\int_{\gamma} f dz = 0$ .

Proof: Let  $a \in \mathbb{C} \setminus G$ , then  $a \notin \gamma$ .

145

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} \frac{f(z)}{z-a} (z-a) dz \\ &= \int_{\gamma} \frac{zf(z)}{z-a} dz - a \int_{\gamma} \frac{f(z)}{z-a} dz \\ &= \int_{\gamma} \frac{zf(z)}{z-a} dz - a \cdot f(a) n(\gamma, a) \\ &= \int_{\gamma} \frac{zf(z)}{z-a} dz - a \cdot f(a) \cdot 0. \end{aligned}$$

Note that  $\frac{zf(z)}{z-a} \rightarrow 0$  as  $a \rightarrow \infty$  uniformly

$$\text{Hence } \int_{\gamma} f(z) dz = \lim_{a \rightarrow \infty} \int_{\gamma} \frac{zf(z)}{z-a} dz = 0.$$

Thm: Under the same hypothesis as to the theorem,

$$\text{we have, } f^{(k)}(a) n(\gamma, a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{k+1}} dz$$

Proof: It follows by the previous lemma.



### Morera's theorem

(144)

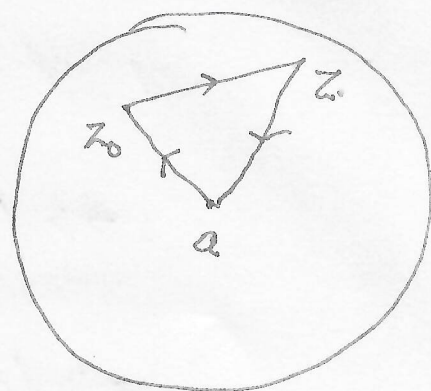
Let  $f$  be a continuous function on domain  $G$  s.t.  $\int f = 0$  for every triangular path  $T$  in  $G$ , then  $f$  is analytic.

Proof: To show  $f$  is analytic on  $G$ , we show  $f$  is analytic on every open disc in  $G$ . Hence, wlog, we assume  $G = B(a, R)$ .

We prove that  $f$  has primitive.

Let

$$F(z) = \int_{[a, z]} f(w) dw$$



Fix a point  $z_0 \in G$ , then by hypothesis

$$\int_{[a, z_0]} f + \int_{[z_0, z]} f + \int_{[z, a]} f = 0$$

$$\Rightarrow F(z) = \int_{[a, z_0]} f + \int_{[z_0, z]} f$$



Here

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0, z]} f(w) dw$$

147

$$\Rightarrow \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \frac{1}{|z - z_0|} \int_{[z_0, z]} |f(w) - f(z_0)| dw$$

Since  $f$  is cont at  $z_0$ , for  $\epsilon > 0$ ,  $\exists \delta > 0$

$$\text{st } |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$

$$|z - z_0| < |z - w| + |w - z_0|$$

$$\text{But } |w - z_0| < \delta \Rightarrow |f(w) - f(z_0)| < \epsilon$$

$$\text{By (*)}, \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| < \epsilon \text{ for } |z - z_0| < \delta.$$

$$\text{i.e. } f'(z_0) = f(z_0).$$