Theorem: Let G be a domain bounded by two closed curves γ_1 and γ_2 s.t. γ_1 lies entirely in the region enclosed by γ_2. If f is analytic on an open set containing G, then

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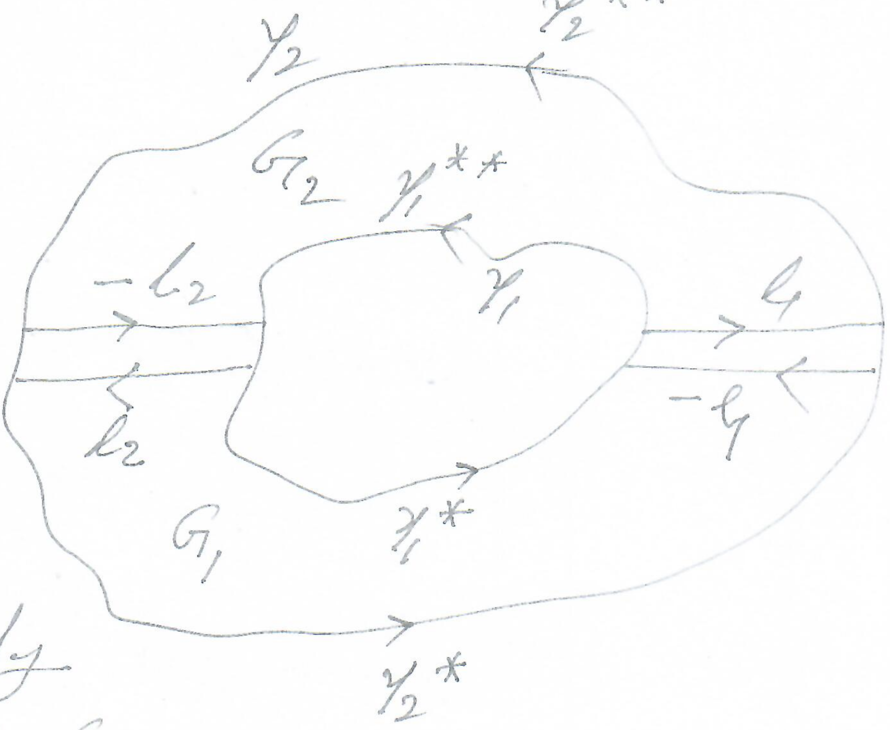
$$\int_{\gamma_1} f(z)\,dz = \int f(z)\,dz$$



Proof: Let

$\Gamma_1 = \gamma_2^* - l_1 - \gamma_1^* + l_2$

$\Gamma_2 = -\gamma_1^{**} + l_1 + \gamma_2^{**} - l_2$

By Cauchy theorem applied to simply conn. domain G_1 & G_2, we get

$$\int_{\Gamma_1} f = 0 = \int_{\Gamma_2} f \Rightarrow \int_{\Gamma_1} f + \int_{\Gamma_2} f = 0$$

$$\Rightarrow \int_{\gamma_1} f = \int_{\gamma_2} f.$$

Application of Morera's theorem

Ex. Suppose f_n is a seq^n of analytic function on $B(0,1)$ s.t. $f_n \to f$ uniformly, then f is analytic on $B(0,1)$.

By Cauchy thm, $\int_\gamma f_n(z)\,dz = 0$

$$\Rightarrow \int_a^b f_n \circ \gamma(t)\, \gamma'(t)\, dt = 0$$

Note that $f_n \circ \gamma \to f \circ \gamma$ uniformly on $[a,b]$ and $\gamma'(t)$ is bdd.

$$\Rightarrow \lim_{n\to\infty} \int_a^b f_n \circ \gamma(t)\, \gamma'(t)\, dt = 0$$

$$\Rightarrow \int_a^b f \circ \gamma(t)\, \gamma'(t)\, dt = 0$$

$$\Rightarrow \int_\gamma f(z)\, dz = 0$$

Hence f is analytic.

Q. Let $h : [a,b] \times B(0,1) \to \mathbb{C}$ be cont. & for each fixed t, $h(t,z)$ is analytic then $H(z) = \int_a^b h(t,z)\, dt$ is analytic on $B(0,1)$.

If $z_n \to z$, then $h(t, z_n) \to h(t, z)$ uniformly because

$$\sup_{a \le t \le b} |h(t, z_n) - h(t, z)| = |h(t_0, z_n) - h(t_0, z)| \to 0.$$

Let γ be any closed smooth curve in $B(0,1)$, then by Cauchy thm

$$\int_\gamma h(t, z)\,dz = 0$$

By Fubini's thm

$$\int_\gamma H(z)\,dz = \int_c^d H \circ \gamma(s)\, \gamma'(s)\,ds$$

$$= \int_c^d \int_a^b h(t, \gamma(s))\, \gamma'(s)\,ds\,dt$$

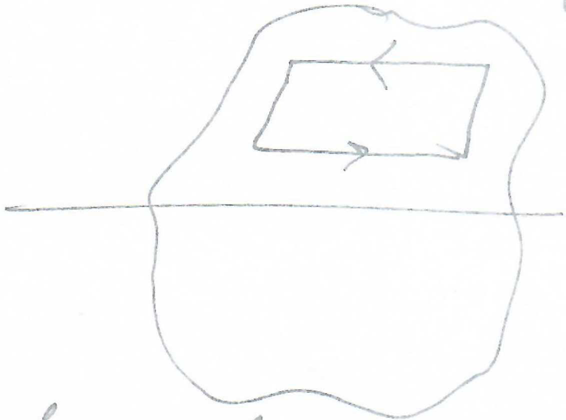
$$= \int_a^b \left(\int_c^d h(t, \gamma(s))\, \gamma'(s) \cancel{ds} \right) d\cancel{t}$$

$$= \int_a^b \left(\int_\gamma h(t, z)\,dz \right) dt$$

$$= 0.$$

Hence H is analytic on $B(0,1)$.

8. Suppose f is cont on a domain G. If f is analytic on $G \setminus \mathbb{R}$, then f is analytic on G. (151)



Suppose R be a closed rectangle contained in G.

Case I. R does not meet real axis, then

$$\int_{\partial R} f = 0.$$

Case II one edge of R lies on the real axis. For $\epsilon > 0$, let

$$R_\epsilon = \{ z \in R : \operatorname{Im}(z) \geq \epsilon \}$$

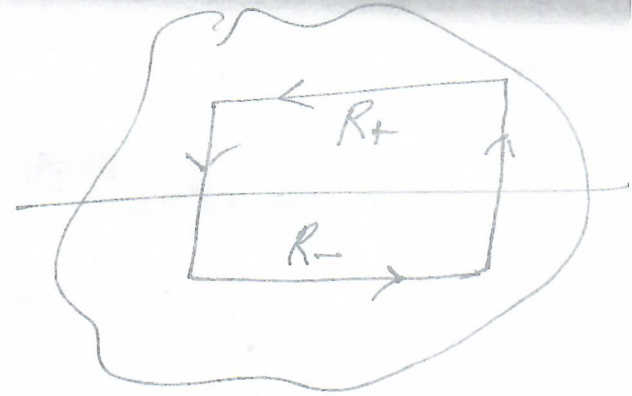
In this case

$$\int_{\partial R} f(z)\,dz = \lim_{\epsilon \to 0} \int_{\partial R_\epsilon} f(z)\,dz = 0.$$

note that integral along the bottom edge is of the form $\int_a^b f(t+i\epsilon)\,dt \to \int_a^b f(t)\,dt$

as $f(t+i\epsilon) \to f(t)$ uniformly as $\epsilon \to 0$.

Case III Top edge of R in the upper half plane and bottom edge in the lower half plane.



Let R_+ be the part of R in the UHP and R_- in the LHP resp. Then

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$$\int_{\partial R} f(z)dz = \int_{\partial R_+} f(z)dz + \int_{\partial R_-} f(z)dz = 0$$

by Case II.

Ex. Let L be a line in the cplx plane. If f is cont. on G and analytic on $G \setminus L$, then f is analytic on G.

~~Corollary~~

Counting zero and open mapping theorem:

We know that if f has a zero at $z = a$, then $\exists$ $m \in \mathbb{N}$ s.t. (153)

$$f(z) = (z-a)^m g(z), \quad g(a) \neq 0$$

and g is analytic, when f was given analytic.

If $a_1, a_2, \ldots a_m$ are zeros of f in G, where some of a_K may repeat according to their multiplicities, then

$$f(z) = (z-a_1)\cdots(z-a_m)\, g(z), \quad g(z) \neq 0$$

for any $z \in G$

By differentiating we get

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \cdots + \frac{1}{z-a_m} + \frac{g'(z)}{g(z)}$$

If γ is a closed rectifiable curve and does not pass any a_K & if $\gamma \approx 0$,

then $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} n(\gamma, a_k)$,

Since $g(z) \neq 0$, $\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$ by Cauchy thm.

If $a_1, \dots, a_m$ are solutions of $f(z)=\alpha$,
then $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)-\alpha}\,dz = \sum_{k=1}^{m} n(\gamma, a_k)$.

Ex. $$\int_{|z|=2} \frac{2z+1}{z^2+z+1}\,dz = \int_{|z|=2} \frac{2z+1}{(z-w_1)(z-w_2)}\,dz,$$
$$= 2\pi i\{1+1\} = 4\pi i,$$
where w_1, w_2 are cubic root of unity.

Proposition: Let $\gamma:[0,1]\to G$ be a closed rectifiable curve in G and $\gamma \approx 0$. Suppose f is analytic in G. Then for $\sigma = f\circ\gamma$, we have
$$n(\sigma,\alpha) = \cancel{\frac{1}{2\pi i}} \sum n(\gamma, a_k),$$
where $a_k \in G$ s.t. $f(a_k)=\alpha$, $\alpha \notin \{\sigma\}$.

Proof: $$n(\sigma,\alpha) = \frac{1}{2\pi i}\int_{\sigma} \frac{dw}{w-\alpha}$$
$$= \frac{1}{2\pi i}\int_0^1 \frac{f'\circ\gamma(t)\,\gamma'(t)\,dt}{f\circ\gamma(t)-\alpha}$$
$$= \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)-\alpha}\,dz = \sum_{k=1}^{m} n(\gamma, a_k).$$

Note that if γ is a closed simple curve enclosing a simply conn. open set G. If γ is freely oriented then (155)

$$n(\gamma, a) = \begin{cases} 1 & \text{if } a \in G \\ 0 & \text{if } a \notin G \end{cases}$$

In this case

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} n(\gamma, a_k)$$

$= $ No of zeros of f in G.

In particular, if $a_1 \dots a_m$ are solutions of $f(z) = \alpha$ in G, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha}\,dz = \text{No of solutions of } f(z) = \alpha \text{ in } G,$$

Also, if $\sigma = f\circ\gamma$, then

$$n(\sigma, \alpha) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w-\alpha}\,dw = \int_\gamma \frac{f'(z)}{f(z)-\alpha}\,dz$$

$= $ no. of solution of $f(z) = \alpha$ in G.

Theorem: Suppose f is analytic in $B(a;R)$ and $\alpha = f(a)$. If $f(z)-\alpha$ has zero of order m at $z=a$, then $\exists\ \epsilon>0$ and $\delta>0$ s.t. for $|\zeta-\alpha|<\delta$, $f(z)-\zeta=0$ has exactly m simple roots in $B(a;\epsilon)$.

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Proof: Since zeros of analytic function are isolated, we can choose $\epsilon>0$ s.t. $\epsilon<\frac{1}{2}R$, $f(z)=\alpha$ has no solution in $0<|z-a|<2\epsilon$

Since $f(z)-f(a) = (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a)+\cdots$

$$\Rightarrow f'(z) = f'(a) + \frac{(z-a)}{1!}f''(a) + \frac{(z-a)^2}{2!}f'''(a)+\cdots$$

it follows that $f'(z)\neq 0$ for $0<|z-a|<2\epsilon$ (if $m\geq 2$, then $f'(a)=0$).

Let $\gamma(t) = a+\epsilon e^{2\pi i t}$, $0\leq t\leq 1$.

Then $\alpha = f(a) \notin \text{range}(\sigma)$, since (157)

$f \circ \gamma(t) = \alpha$ has no solution in

$0 < |z-a| < 2\epsilon$. Therefore, $\exists\ \delta > 0$

s.t. $B(\alpha, \delta) \cap \text{range}(\sigma) = \Phi$

Note that $B(\alpha, \delta)$ is contained in a component of $\mathbb{C} \setminus \{\sigma\}$ enclosed by σ.

Thus, for $|\alpha - \xi| < \delta$, we get

$$n(\sigma, \alpha) = n(\sigma, \xi) = \sum_{n=1}^{p} n(\gamma, z_k(\xi))$$

But $n(\gamma, z)$ must be either 0 or 1,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - \xi}\, dz = \sum_{n=1}^{p} n(\gamma, z_k(\xi))$$

$$= \sum_{n=1}^{m} n(\gamma, z_k(\xi))$$

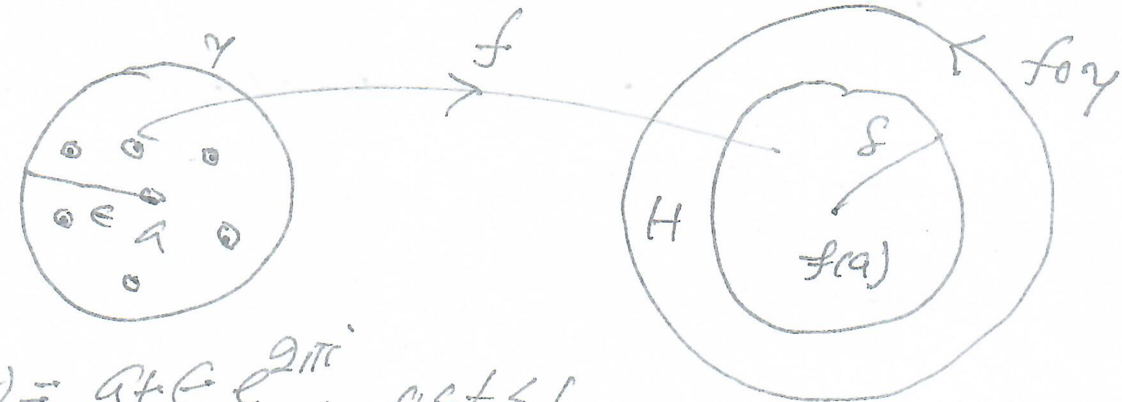
Hence, $f(z) - \xi$ has exactly m solutions inside $B(a, \epsilon)$. Since $f'(z) \neq 0$ for $0 < |z - a| < \epsilon$, each of these for $\xi \neq \alpha$ must be simple.

Open mapping theorem:

Let f be a non-constant analytic function on domain G. Then for any open set $U \subset G$, $f(U)$ is open.

(158)

Proof: It is enough to show that for any disc $B(a,\epsilon) \subset G$, $f(B(a,\epsilon))$ is open.



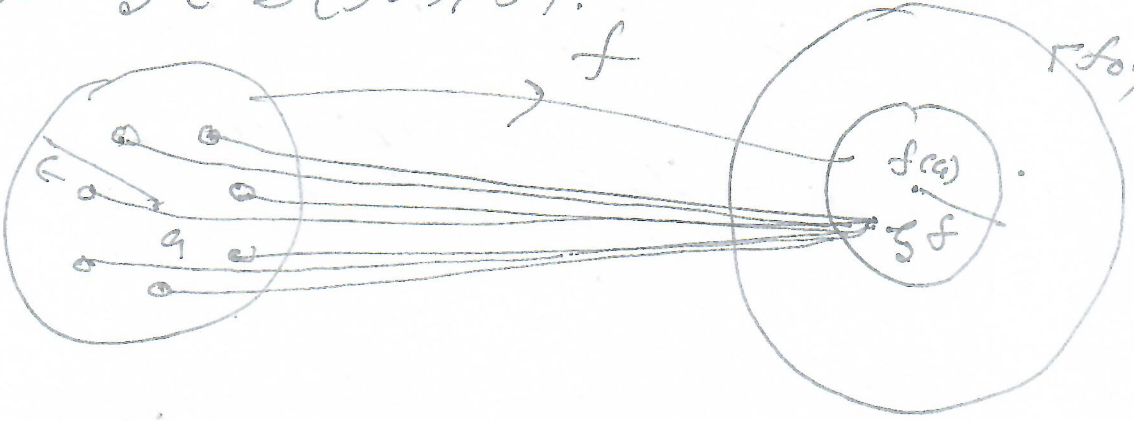
Let $\gamma(t) = a + \epsilon e^{2\pi i t}$, $0 \le t \le 1$.

Then $$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-f(a)}\, dz = n \geq 1.$$

Now, $f(a) \in H$, an open & conn. component of $\mathbb{C} \setminus \{f\circ\gamma\}$, we can choose $\delta > 0$ st. $B(f(a),\delta) \subset H$. Then for $z \in \{\gamma\}$, and $\zeta \in B(f(a),\delta)$

$\Rightarrow f(z) - \zeta \neq 0$.

Hence
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\xi}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-\alpha} = n. \quad (159)$$
$$\forall\ \xi \in B(f(a),\delta)$$
That is $f(z)=\xi$ has n solutions for each $\xi \in B(f(a),\delta)$.



Ex. If $f: G \to \mathbb{C}$ is analytic and one-one, then $f'(z) \neq 0$ for any $z \in G$.

Suppose $\exists\ z_0 \in G$ s.t. $f'(z_0)=0$.

Then for $g(z) = f(z+z_0) - f(z_0)$
$$g(0)=0,\quad g'(0) = f'(z_0) = 0.$$
(because translation does not alter injectivity)

Hence w.l.g. we can assume
$$f(0) = f'(0) = 0.$$

Then $f(z) = z^k g(z)$, $g(0) \neq 0$. (160)

Since $g(z) \neq 0$ for $z \in B(0,\epsilon)$,
we can find h analytic s.t.

$$(h(z))^k = g(z)$$

($\because$ $k \operatorname{Log} h(z) = \operatorname{Log} g(z)$

$$\Rightarrow h(z) = \exp\left(\frac{1}{k} \operatorname{Log} g(z)\right)$$

Now $f(z) = (z\,h(z))^k$

Since $\varphi(z) = z\,h(z)$ is analytic in $B(0,\epsilon)$, by open mapping thm,

$\varphi(B(0,\epsilon))$ is open

Hence $\exists\ \delta > 0$ s.t.

$$B(0, 2\delta) \subset \varphi(B(0,\epsilon))$$

choose $z_1, z_2 \in B(0,\epsilon)$ s.t.

$$\varphi(z_1) = \delta,\ \varphi(z_2) = \delta \exp\left(\frac{2\pi i}{k}\right)$$

Then $f(z_2) = \delta^k \left(\exp\left(\frac{2\pi i}{k}\right)\right)^k = \delta^k = f(z_1)$.

$\Rightarrow$ f is not one-one

Singularities

(16)

A point $z_0 \in \mathbb{C}$ is called singular point or singularity of f if f is not analytic at z_0, but each nbd of z_0 contains a point where f is analytic.

Ex. $\frac{e^z - 1}{z}, \frac{1}{z^2}, \sin\frac{1}{z}, \text{Log } z$ has singularity at $z=0$.

Primarily, a singularity can be classified into two ways:

(i) A singular point z_0 is said to be isolated singularity or isolated singular point of f if f is analytic in $B(z_0, \epsilon) - \{z_0\}$ for some $\epsilon > 0$.

(ii) A singular point is said to be non-isolated singularity of if not isolated singularity.

Ex. $\frac{\sin z}{z}$, $\frac{1}{z^2}\sin\frac{1}{z}$; 0 is an isolated singularity.

Ex: $\frac{1}{\sin \pi/z}$, $\operatorname{Log} z$ have non-isolated singularity at $z=0$.

For $\frac{1}{\sin \pi/z}$; $\frac{\pi}{z_n} = n\pi$, $z_n = \frac{1}{n} \to 0$

Removable singularity:

An isolated singular point $z_0 \in \mathbb{C}$ is said to be removable singularity if $\left|\lim_{z\to z_0} f(z)\right|$ is finite.

Ex. $f(z) = \frac{\sin z}{z}$, $\lim_{z\to 0}\frac{\sin z}{z} = 1$.

Let $\lim_{z\to z_0} f(z) = l$.

Define $g(z) = \begin{cases} f(z) & \text{if } z \neq z_0 \\ l & \text{if } z = z_0. \end{cases}$

Hence g is analytic at $z=z_0$.

Laurent Series:

For $0<r<R$, let f be analytic on annulus.

$$\mathrm{ann}(a, r, R)=\{z : r<|z-a|<R\}.$$

then for each $z \in \mathrm{ann}(a, r, R)$, f has Laurent Series representation

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-a)^n \longrightarrow (*)$$

where the Series in the RHS Converges absolutely and uniformly on every $\overline{\mathrm{ann}}(a, r_1, R_1)$ with $r<r_1<R_1<R$.

The Coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-a)^{n+1}}\, dz$$

where $\gamma(t)=a+s e^{2\pi i t}, 0\le t\le 1, r<s<R.$

This series is unique and $\sum_{n=-\infty}^{-1} a_n (z-a)^n$

is called principal part and (164)

$\sum_{n=0}^{\infty} a_n (z-a)^n$ is called regular part.

or analytic part.

Ex. $f(z) = \frac{1}{1-z}$. for $|z|<1$,

$f(z) = 1 + z + z^2 + \dots$

whereas for $|z|>1$, $\left|\frac{1}{z}\right| < 1$.

$f(z) = -\frac{1}{z} \frac{1}{\left(1-\frac{1}{z}\right)} = -\frac{1}{z} - \frac{1}{z^2} - \dots$

Ex. $f(z) = \frac{\sin z}{z} = \frac{1}{z} \sum \frac{(-1)^n z^{2n+1}}{(2n+1)!}$

$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$

Ex. $f(z) = \frac{e^z - 1}{z^3} = \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{6} + \frac{z}{24} + \dots$

Ex. ~~If~~ f is an entire function and $\lim_{|z|\to\infty} \left|\frac{f(z)}{z}\right| = 0$, then f is constant

$$\frac{f(z)}{z} = \frac{a_0}{z} + a_1 + a_2 z^2 + \cdots \infty$$

$$\lim_{z\to\infty}\left|\frac{f(z)}{z}\right| = 0 \Rightarrow a_1 = 0,\ a_2 = 0 \cdots$$

$$\Rightarrow f(z) = a_0.$$

<u>Result</u>: TFAE

(i) f has removable singularity at $z=z_0$.

(ii) $\forall\, a_{-n} = 0$, in Laurent series exp^n of f about z_0.

(iii) $\lim_{z\to z_0} (z-z_0) f(z) = 0$

(iv) f is bounded in a deleted nbd of z_0.

Proof: We have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} \quad \text{—} (*)$$

$\left|\lim_{z\to z_0} f(z)\right|$ is finite iff $a_{-n} = 0$;

$\Rightarrow$ (i) $\Leftrightarrow$ (ii).

$\Rightarrow$ (i) $\Leftrightarrow$ (ii).

Since the series (*) converges uniformly

$$\lim_{z \to z_0} (z-z_0) f(z) = 0 \Leftrightarrow a_{-1} = a_{-2} = \cdots = 0$$

Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(166)

Next we show that

(iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (iv).

It follows from Laurent series expansion.

A pole is an isolated singular point $z_0 \in \mathbb{C}$ s.t.

$$\left|\lim_{z \to z_0} f(z)\right| = \infty.$$

That is, $\forall M > 0$, $\exists \delta > 0$ s.t.

$$|f(z)| > M \text{ for } 0 < |z - z_0| < \delta.$$

$$\Rightarrow \left|\frac{1}{f(z)}\right| < M \text{ for } 0 < |z - z_0| < \delta.$$

i.e. $\frac{1}{f(z)}$ is bounded in a deleted

nbd of z_0. Hence $\frac{1}{f(z)}$ has removable singularity at z_0.

Define $h(z) = \begin{cases} \dfrac{1}{f(z)} & z \neq z_0 \\ 0 & z = z_0 \end{cases}$

Since $\lim_{z \to z_0} \dfrac{1}{f(z)} = 0$.

Then h is analytic in $B(z_0, R)$ for some $R > 0$. Also, $h(a) = 0$, so let

$\dfrac{1}{f(z)} = h(z) = h_1(z)(z - z_0)^n$, $h_1(z_0) \neq 0$

$\Rightarrow (z-a)^n f(z) = (h_1(z))^{-1}$

$\lim (z-a)^n f(z) = (h_1(a))^{-1} \neq 0.$

Result: FAE

(i) f has pole of order m at z_0

(ii) $f(z) = \dfrac{g(z)}{(z-z_0)^m}$ g is analytic at z_0 & $g(z_0) \neq 0$.

(iii) $\dfrac{1}{f}$ has zero of order m

Result: Limit point of zeros of an analytic function on domain G is essential singularity if the limit itself an isolated singular point. (169)

Proof: Let $z_n \to z_0$ & $f(z_n) = 0$.

If z_0 is a removable singularity of f, then f is cont at z_0 and z_0 is not a singular pt.

On the other hand, suppose z_0 is a pole of f. Then

$$\lim_{z \to z_0} f(z) = \infty$$

i.e $|f(z)| > M > 0$ for $0 < |z - z_0| < \delta$.

which is not possible as each ~~nbd~~ the deleted nbd $0 < |z - z_0| < \delta$ contains ~~∞~~ infinitely many zeros of f.

Hence, z_0 is an essential singularity.

(iv) $\lim_{z \to z_0} (z-z_0)^{m+1} f(z) = 0$

(v) $(z-z_0)^m f(z)$ has removable singularity.

Proof: (i) $\Leftrightarrow$ (ii) has been work out just before. Other implications follows easily.

* An isolated singular point is said to be essential singularity if it is neither removable nor pole.

Ex. $f(z) = e^{1/z}$, $z=0$ is an essential singularity.

Result: FAE

(i) f has essential singularity at $z = z_0$

(ii) $\lim_{z \to z_0} f(z)$ does not exist

(iii) infinitely many $a_{-n} \neq 0$.

Result: Let z_0 be the limit of poles z_n of f. Since each nbd of z_0 contains infinitely many poles, z_0 cannot be removable singularity. Also, poles are isolated, it cannot be a pole. Hence z_0 is a non-isolated essential singularity. (17.2)

Ex. $f(z) = \sin\frac{1}{z}$, $\frac{1}{z_n} = n\pi$, $z_n = \frac{1}{n\pi}$ are zeros of f & $z_n \to 0 = z$.
$z=0$ is ess. singularity of f.

Casorati–Weierstrass thm:

If f has essential singularity at $z=a$, then for any $\delta > 0$,

$$\overline{f[\mathrm{ann}(a, 0, \delta)]} = \mathbb{C}$$

Proof: Suppose f is analytic in ann$(a;0,R)$. (17)

We need to claim that if $c \in \mathbb{C}$ & $\epsilon > 0$, then $\forall\, \delta > 0$, $\exists\, z \in \text{ann}(a;0,\delta)$ s.t. $|f(z)-c| < \epsilon$.

Assume the above claim is false.

That is, $\exists\, c \in \mathbb{C}$ & $\epsilon > 0$, s.t.

$$|f(z)-c| \geq \epsilon, \quad \forall\, z \in \text{ann}(a;0,\delta)$$

Then $\lim |z-a|^{-1}|f(z)-c| = \infty$

~~$\Rightarrow f$ has pole at $z=a$.~~

s.t. m is the order of this pole of $(z-a)^{-1}(f(z)-c)$. Then

$$\lim_{z \to a} |z-a|^{m+1}|f(z)-c| = 0$$

$$\Rightarrow |z-a|^{m+1}|f(z)| \leq \underbrace{|z-a|^{m+1}|f(z)-c| + |z-a|^{m+1}|c|}_{\to 0}$$

$(z-a)^m f(z)$ has removable singularity.

which implies $z=a$ is not ess. singularity of f.

(172)

Let $g(z)=f(\frac{1}{z})$. Then the nature of singularity of f at ∞ can be understood by nature of singularity of g at 0.

Ex. $f(z)=z^3$ has pole of order 3 at ∞.

Ex. e^z has ess. singularity at $z=\infty$.

Ex. An entire function f has a removable at ∞ iff f is constant.

Ex. An entire function has pole of order m at ∞ iff f is a poly of degree m.

$$f(z)=\sum a_n z^n$$

$$f(\tfrac{1}{z})=\sum_{n=0}^{\infty}\frac{a_n}{z^n}=\sum_{n=0}^{m}\frac{a_n}{z^n}\Rightarrow f(z) \text{ is poly.}$$

Residues:

Let f has an isolated singularity at $z = a$, and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$ (173)

be its Laurent's series expansion of f about a.

Then residue of f at $z = a$ is the coeff. a_{-1}. We denote it by

$$Res(f, a) = a_{-1}$$

Residue theorem:

Let f be analytic in a domain G except for isolated singular points $a_1 \dots a_m$. Let γ be a closed rectifiable curve in G which does not pass through any of the point a_k & if $\gamma \approx 0$ in G, then

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{m} n(\gamma, a_k)\, Res(f, a_k)$$

Proof: Let $m_k = n(\gamma, a_k)$, $1 \le k \le m$ (174)

$\bar{B}(a_k, r_k) \cap \{\gamma\} = \Phi$.

$\bar{B}(a_k, r_k) \cap \bar{B}(a_j, r_j) = \Phi$ for $k \ne j$.

Let $\gamma_k(t) = a_k + r_k e^{-2\pi i m_k t}$, $0 \le t \le 1$.

$n(\gamma, a_j) + \sum_{k=1}^{m} n(\gamma_k, a_j) = 0$

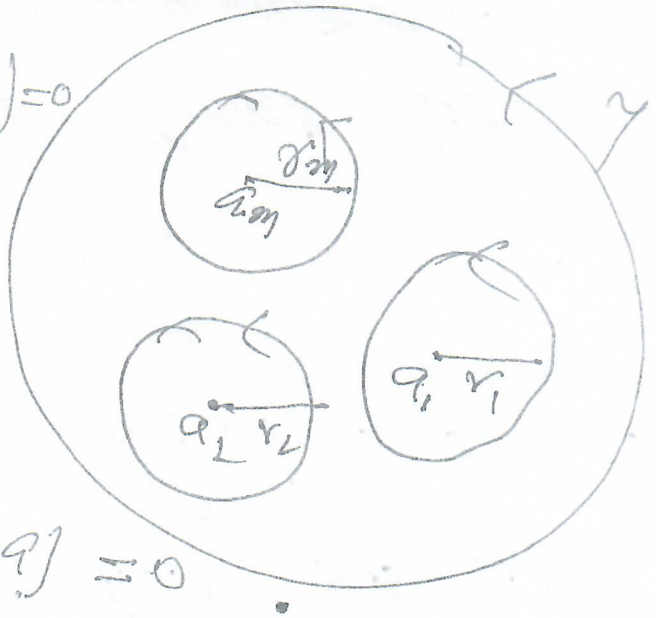
Since $\gamma \approx 0$ in G,

and

$\overline{\bar{B}(a_k, r_k)} \subset G$

$n(\gamma, a) + \sum_{k=1}^{m} n(\gamma_k, a) = 0$

for all $a \notin G \setminus \{a_1, \ldots a_m\}$.



Since f is analytic in $G - \{a_1, \ldots a_m\}$

by Cauchy theorem for multiply conn. domain,

$$\int_\gamma f + \sum_{k=1}^{m} \int_{\gamma_k} f = 0.$$

Now, $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a_k)^n$ converges uniformly on $\overline{B(a_k, r_k)}$, it follows that

(175)

$$\int_{\gamma_k} f = \sum_{-\infty}^{\infty} b_n \int_{\gamma_k} (z-a_k)^n$$

$$= 2\pi i \, n(\gamma_k, a_k) \cdot Res(f, a_k)$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\gamma} f = \sum_{k=1}^{m} n(\gamma_k, a_k) Res(f, a_k).$$

If f has pole of order m at $z = a$, then

$g(z) = (z-a)^m f(z)$ has removable singularity at $z = a$, $g(a) \neq 0$.

Let $g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$ be the power series of g about a. Then

$$f(z) = \frac{b_0}{(z-a)^m} + \cdots + \frac{b_{m-1}}{(z-a)} + \sum_{k=0}^{\infty} b_{m+k} (z-a)^k$$

$\Rightarrow \operatorname{Res}(f, a) = b_{m-1} = \frac{1}{(m-1)!} g^{(m-1)}(a).$ (178)

For particular, if $z = a$ is a simple pole then

$$\operatorname{Res}(f, a) = b_0 = g(a) = \lim_{z \to a} (z-a) f(z)$$

ex. $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} \, dx = \frac{\pi}{\sqrt{2}}$

$f(z) = \frac{z^2}{1+z^4}$; $1+z^4 = 0$

$\rightarrow f$ has poles at $e^{i\theta}$; $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

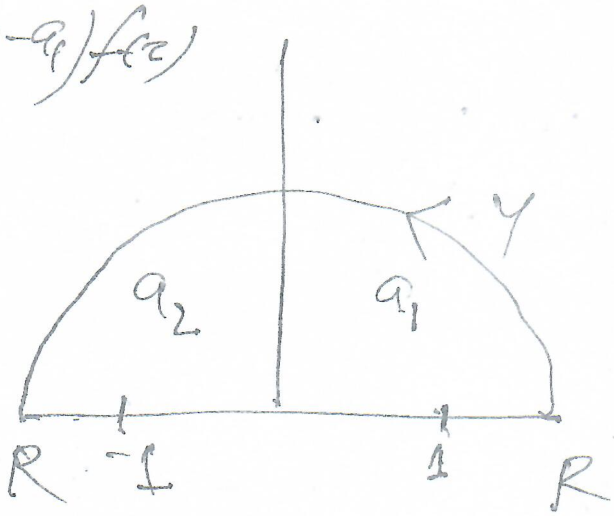
Let $a_n = \exp\left[i \left\{ \frac{\pi}{4} + (n-1)\frac{\pi}{2} \right\} \right]$

$$\operatorname{Res}(f, a_1) = \lim_{z \to a_1} (z - a_1) f(z)$$

$$= \frac{1-i}{4\sqrt{2}}$$

Similarly

$\operatorname{Res}(f, a_2) = \frac{-1-i}{4\sqrt{2}}$.



$$\frac{1}{2\pi i}\int_\gamma f = \text{Res}(f, q_1) + \text{Res}(f, q_2) \qquad (47)$$

$$= \frac{-i}{2\sqrt{2}}.$$

Now,

$$\frac{1}{2\pi i}\int_\gamma f = \frac{1}{2\pi i}\int_{-R}^{R} \frac{x^2}{1+x^4}\,dx + \frac{1}{2\pi i}\int_0^{\pi} \frac{R^3 e^{3it}}{1+R^4 e^{4it}}\,dt$$

Note that

$$\left|\frac{1}{2\pi i}\int_0^{\pi}\left(\frac{R^3 e^{3it}}{1+R^4 e^{4it}}\right)dt\right|$$

$$\leq \frac{1}{2\pi}\int_0^{\pi} \frac{R^3}{R^4-1}\,dt \longrightarrow 0 \text{ as } R \to \infty$$

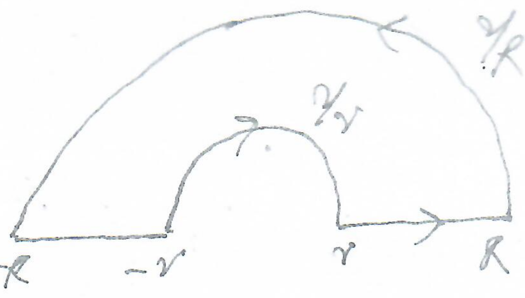
Hence

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}.$$

Show that $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx = \pi/2.$

Let $f(z) = \dfrac{e^{iz}}{z}$. Then f has a simple pole at $z=0$. Let $0 < r < R$

By Cauchy theorem



$$0 = \int_\gamma f(z)\,dz.$$

$$0 = \int_{\gamma}^{R} \frac{e^{ix}}{x}\,dx + \int_{\gamma_R} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\gamma} \frac{e^{ix}}{x}\,dx + \int_{\gamma_\gamma} \frac{e^{iz}}{z}\,dz \quad \text{---}(*)$$

Now, $\int_{\gamma}^{R} \frac{\sin x}{x}\,dx = \frac{1}{2i} \int_{\gamma}^{R} \frac{e^{ix} - e^{-ix}}{x}\,dx$ (178)

$$= \frac{1}{2i} \int_{\gamma}^{R} \frac{e^{+ix}}{x}\,dx + \frac{1}{2i} \int_{-R}^{-\gamma} \frac{e^{+ix}}{x}\,dx.$$

Now, $\left| \int_{\gamma_R} \frac{e^{iz}}{z}\,dz \right| \leq \int_{0}^{\pi} e^{-R\sin\theta}\,d\theta \to 0$ as $R \to \infty$ $\quad$ —$(**)$

Since $\frac{e^{iz}-1}{z}$ has removable singularity at $z=0$, $\exists\, M>0$ s.t.

$$\left| \frac{e^{iz}-1}{z} \right| \leq M \quad \text{for } |z| \leq 1$$

$$\left| \int_{\gamma_\gamma} \frac{e^{iz}-1}{z}\,dz \right| \leq \pi \gamma M \to 0.$$

But $\int_{\gamma_\gamma} \frac{1}{z}\,dz = -\pi i$ for each $\gamma > 0$

$$\Rightarrow -\pi i = \lim_{\gamma \to 0} \int_{\gamma_\gamma} \frac{e^{iz}}{z}\,dz. \quad \text{---}(***)$$

By Combining we get $\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \pi/2.$

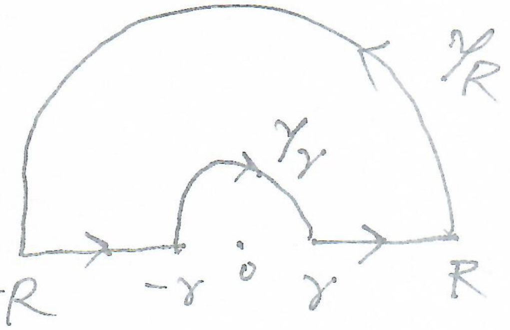
Ex. Show that $\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$ (179)

For $z = |z|e^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{5\pi}{2}$,

$$\operatorname{Log} z = \log|z| + i\theta$$

note that if $x > 0$, then $\operatorname{Log} x = \log x$, if $x < 0$, then $\operatorname{Log} x = \log|x| + \pi i$

$$\int_\gamma \frac{\operatorname{Log} z}{1+z^2}\,dz = \int_r^R \frac{\log x}{1+x^2}\,dx$$



$$+ iR\int_0^\pi \frac{[\log R + i\theta]}{1+R^2e^{2i\theta}}e^{i\theta}\,d\theta$$

$$+ \int_{-R}^{-r} \frac{\log|x| + \pi i}{1+x^2}\,dx + ir\int_\pi^0 \frac{\log r + i\theta}{1+r^2e^{2i\theta}}e^{i\theta}\,d\theta$$

since the only pole of $\frac{\operatorname{Log} z}{1+z^2}$ inside γ is at $z = i$, the residue of it at $z=i$

$$= \frac{1}{2i}\left[\log|i| + \frac{1}{2}\pi i\right] = \pi/4$$

$$\Rightarrow \int_\gamma \frac{\operatorname{Log} z}{1+z^2} = \frac{\pi^2 i}{2}$$

now, $\int_{r}^{R} \frac{\log x}{1+x^2}\,dx + \int_{-R}^{r} \frac{\log|x|+\pi i}{1+x^2}\,dx = 2\int_{r}^{R} \frac{\log x}{1+x^2} + \pi i \int_{r}^{R} \frac{dx}{1+x^2}$

letting $r \to 0$ & $R \to \infty$ and using the fact that $\int_0^\infty \frac{dx}{1+x^2} = \pi/2$, (180)

it follows that

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx = \frac{1}{2} \lim_{r\to 0} ir \int_0^{\pi} \left(\frac{\log r + i\theta}{1+r^2 e^{2i\theta}}\right) e^{i\theta}\,d\theta$$

$$- \frac{1}{2} \lim_{R\to\infty} iR \int_0^{\pi} \frac{\log R + i\theta}{1+R^2 e^{2i\theta}} e^{i\theta}\,d\theta$$

$$\cancel{\to 0 \text{ as } r \to 0}$$

$$= 0.$$

Ex. Show that $\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}$

For $z = e^{i\theta}$, $\bar{z} = \frac{1}{z}$

$$a + \cos\theta = a + \frac{1}{2}(z+\bar{z}) = \frac{2az + z^2 + 1}{2z}$$

Please complete it.

Meromorphic function:

A function f on G (open set) is said to be meromorphic if f is analytic on G except possibly on poles. (181)

Argument principle:

Let f be a meromorphic function on an open set G with zeros $z_1, \dots z_n$ and poles $p_1, \dots p_m$ in G. Let γ be a closed rectifiable curve in G with $\gamma \approx 0$, and not passing through any of p_j, z_k. Then

$$\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} n(\gamma, z_k) - \sum_{j=1}^{m} n(\gamma, p_j)$$

Proof: If f has a zero of order m at $z=a$, then $f(z) = (z-a)^m g(z)$, $g(a) \neq 0$, g is analytic near a.

Hence $\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$

Now, suppose f has pole of order m at $z=a$, then

$$f(z) = (z-a)^{-m} g(z), \quad g \text{ is analytic}$$

near a and $g(a) \neq 0$. (182)

Then $$\frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{g'(z)}{g(z)}.$$

By repeated use of these two facts, we get

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z-a_k} - \sum_{j=1}^{m} \frac{1}{z-b_j}$$

Since $\frac{g'(z)}{g(z)}$ is analytic, by Cauchy

then $\int_\gamma \frac{g'(z)}{g(z)} dz = 0$. Hence

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} n(\gamma, a_k) - \sum_{j=1}^{m} n(\gamma, b_j)$$

Cor^n. If γ is a simple closed rectifiable curve, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Z - P$$

where Z = no. of zeros & P = no. of poles.

Ex: If in addition g is analytic in G, then (183)

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} g(z_k)\, n(\gamma, z_k) - \sum_{j=1}^{m} g(p_j)\, n(\gamma, p_j)$$

Note that a one-one analytic function f has inverse. It is remarkable that the above theorem can be used to get inverse.

Suppose $R>0$, and f is analytic, one-one on $\overline{B(a;R)}$. Let $\Omega = f(B(a;R))$.

If $|z-a|<R$, and $\xi = f(z) \in \Omega$, then $f(w)-\xi$ has only one solution in $B(a;R)$. If we take $g(w)=w$ in the above theorem, then

$$z = \frac{1}{2\pi i}\int_\gamma \frac{w f'(w)}{f(w)-\xi}\,dw.$$

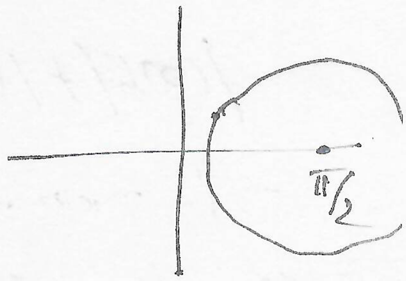
where γ is the circle $|z-a|=R$.

Put $z = f^{-1}(\xi)$. Then we have the following proposition. (184)

Prop: Let f be analytic on an open set containing $\overline{B(a,R)}$ and f is one-one on $B(a,R)$. If $\Omega = f(B(a,R))$ and γ is $|z-a|=R$, then $f^{-1}(w)$ for each $w \in \Omega$ is given by formula

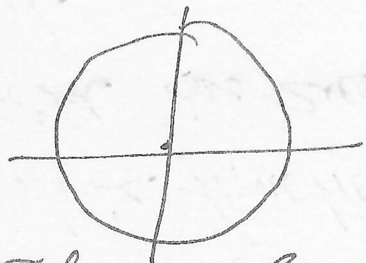
$$f^{-1}(w) = \frac{1}{2\pi i}\int_{\gamma} \frac{z f'(z)}{f(z)-w}\,dz .$$

*Evaluate $\displaystyle\int_{|z-\frac{\pi}{2}|=1} \tan z \, dz = -\int_{|z-\pi/2|=1} \frac{d(\cos z)}{\cos z}\,dz$



* $\displaystyle\int_{|z|=1} \frac{dz}{\sin z} = \int \frac{f'(z)}{f(z)}\,dz$

where $f(z) = \tan z/2$.



(Both cases use argument principle.

Rouche's theorem:

Let f and g be meromorphic functions in a nbd of $\overline{B}(a,R)$ with no zero and pole on $|z-a|=R$. If $Z_f, Z_g (P_f, P_g)$ are no. of zeros (poles) of f and g resp. and if

$$|f(z)+g(z)| < |f(z)|+|g(z)|$$

on $|z-a|=R$, then

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$$Z_f - P_f = Z_g - P_g$$

Proof: We have $\left|\frac{f(z)}{g(z)}+1\right| < \left|\frac{f(z)}{g(z)}\right|+1$ on $\gamma(t) = a+Re^{it}$.

If $\lambda = \frac{f(z)}{g(z)} \in [0,\infty)$, then

$$\lambda+1 < \lambda+1 \quad \text{a contradiction}$$

Hence f/g maps γ onto $\Omega = \mathbb{C} \setminus [0,\infty)$. (?)

If l is a branch of the logarithm on Ω, then

$$\left(\ln\left(\frac{f(z)}{g(z)}\right)\right)' = \frac{(f/g)'}{f/g} \quad \text{in a nhd of } \gamma.$$

Then $0 = \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{f/g} = \frac{1}{2\pi i}\int \frac{f'g - fg'}{fg}$

$$\Rightarrow \frac{1}{2\pi i}\int \frac{f'}{f} - \frac{1}{2\pi i}\int \frac{g'}{g} = 0$$

$\Rightarrow$ ~~$Z_f - P_f$~~

$$Z_f - P_f = Z_g - P_g.$$

Cor: If $P_f = P_g = 0$, then f & g have same no. of zero in $B(a,R)$.

Cor: If $|g(z)| < |f(z)|$ for $z \in \gamma$, then $Z_f = Z_{f+g}$.

Ex. Find the no. of zeros of $z^7 - 4z^3 + z - 1 = 0$ in $|z| < 1$.

Let $f(z) = -4z^3$, $g(z) = z^7 + z - 1$.

Then $|f(z)| = 4$, $|g(z)| \le 3$ on $|z|=1$

$\Rightarrow |g(z)| < |f(z)|$.

Hence no. of zeros of f is 3 in $|z|<1$.

<u>Schwarz's Lemma</u>:

Suppose f is analytic in $D = \{z : |z|<1\}$ with

(a) $|f(z)| \le 1$ for $z \in D$

(b) $f(0) = 0$

Then $|f'(0)| \le 1$ and $|f(z)| \le |z|$, $\forall z \in D$.

Moreover, if $|f'(0)| = 1$ or if $|f(z_0)| = |z_0|$ for some $z_0 \in D$, then $\exists$ $c \in \mathbb{C}$ s.t. $|c| = 1$ and $f(z) = cz$.

Proof: Define $g : D \longrightarrow \mathbb{C}$ by

$$g(z) = \begin{cases} f(z)/z & \text{if } z \neq 0 \\ f'(0) & \text{if } z = 0 \end{cases}$$

Then g is analytic and by MMT,

$$|g(z)| \le \left|\frac{f(z_0)}{z_0}\right| \le \frac{1}{r} \text{ for } |z| \le r, \ 0 < r < 1.$$

Letting $r \to 1$, we get $|g(z)| \le 1$ $\forall z \in D$

i.e. $|f(z)| \le |z|$ for $z \in D$.

and $|f'(0)| = |g(0)| \le 1$. (188)

If $|f(z_0)| = |z_0|$ for some $z_0 \in D$; $z_0 \neq 0$ or $|f'(0)| = 1$, then $|g|$ attains its maximum value inside $D \Rightarrow g(z) = c$

But $|g(0)| = |f'(0)| = 1 \Rightarrow |c| = 1$

(or $\left|\frac{f(z_0)}{z_0}\right| = 1 \Rightarrow |c| = 1$)

We will apply Schwarz's lemma to characterise the conformal maps from the disc D onto D.

For $|a| < 1$, define $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$.

Then φ_a is analytic for $|z| < \frac{1}{|a|}$.

that is, φ_a is analytic in an open disc containing D.

Also, $\varphi_a(\varphi_{-a}(z)) = z = \varphi_{-a}(\varphi_a(z))$ for $|z| < 1$.

Hence φ_a maps unit disc onto itself.

For θ real, we have (189)

$$|\varphi_a(e^{i\theta})| = \left|\frac{e^{i\theta}-a}{1-\bar{a}e^{i\theta}}\right| = \left|\frac{e^{i\theta}-a}{\overline{e^{i\theta}-a}}\right| = 1.$$

$\Rightarrow \varphi_a(\partial D) = \partial D.$

Also, it is easy to see that

$$\varphi_a'(0) = 1-|a|^2, \quad \varphi_a'(a) = \frac{1}{1-|a|^2}$$

Now, use Schwarz's lemma to characterise analytic function f from D with $|f(z)| \leq 1$. Also suppose $|a|<1$, $f(a) = \alpha$ (then $|\alpha|<1$)

Among all functions f having these properties, what is the maximum of $|f'(a)|$?

To show this, let $g = \varphi_\alpha \circ f \circ \varphi_{-a}$

then $g : D \to D$ and

$$g(0) = \varphi_\alpha(f(a)) = \varphi_\alpha(\alpha) = 0.$$

Then by Schwarz's Lemma $|g'(0)| \le 1$. (190)

By applying chain rule to g, we get

$$g'(0) = (\varphi_\alpha \circ f)'(\varphi_a(0))\,\varphi_a'(0)$$
$$= (\varphi_\alpha \circ f)'(a)(1-|a|^2)$$
$$= \varphi_\alpha'(f(a))\,f'(a)\,(1-|a|^2)$$
$$= \frac{(1-|a|^2)\,f'(a)}{1-|\alpha|^2}$$

Then $|f'(a)| \le \dfrac{1-|\alpha|^2}{1-|a|^2}$. —— (1)

Equality holds if $|g'(0)| = 1$ or by Schwarz's Lemma, $\exists\, c$ with $|c| = 1$

s.t. $\quad g(z) = cz \quad \forall z \in D$ —— (2)

put $z = \varphi_a(z)$, then

$$\varphi_\alpha \circ f \circ \varphi_{-a} \circ \varphi_a(z) = c\,\varphi_a(z)$$

$$f(z) = \varphi_{-\alpha}(c\,\varphi_a(z)) \text{ —— (3)}$$

for $|z| < 1$.

Note that if $|c| = 1$ & $|\alpha| < 1$, then $f = c\varphi_\alpha$ defines a bijection on D

In addition, the converse of this is also true. (191)

<u>Thm</u>: Let $f : D \longrightarrow D$ be an analytic bijection, $f(a) = 0$. Then $\exists\ c \in \mathbb{C}$ with $|c| = 1$ s.t $f = c\varphi_a$.

Proof: Since $f : D \to D$ is analytic bijection, $\exists\ g : D \longrightarrow D$ s.t

$$g \circ f(z) = z \quad \text{for } |z| < 1.$$

By chain rule

$$g'(f(a)) f'(a) = 1$$

$$g'(0) f'(a) = 1 \qquad \text{———— } (4)$$

By (1), $|f'(a)| \leq \frac{1}{1-|a|^2}$ ——— (5)

$$|g'(0)| \leq 1 - |a|^2 \qquad \text{——— } (6)$$

From (4), (5) & (6), it follows that

$$|f'(a)| = \frac{1}{1-|a|^2}$$

That is, equality occurs in (5); hence $\exists\ c \in \mathbb{C}$ with $|c| = 1$ s.t

$$f(z) = \varphi_{-0}(c\,\varphi_a(z)) = c\,\varphi_a(z).$$

Ex. For $|f(z)| \leq 1$, when $|z| < 1$, define

$g : D \to D$ by

$$g(z) = f \circ \varphi_a(z) = \frac{f(z) - a}{1 - \bar{a} f(z)}.$$

(Hence f is analytic)

Show that $\dfrac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \dfrac{|f(0)| + |z|}{1 + |f(0)||z|}.$