DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA547: Complex Analysis — EndSem
Instructor: Rajesh Srivastava — April 28, 2025
Time duration: Three hours — Maximum Marks: 50

N.B. Answer without proper justification will attract zero mark.

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1. (a) Is every polynomial of degree at least one surjective from $\mathbb{C}$ to itself? **[1]**

   (b) Is it possible for a non-constant entire function f to have infinitely many zeros in every closed ball? **[1]**

   (c) Evaluate $\int_{|z-1|=2} |z+1|^2 dz$ using residue theorem. **[1]**

   (d) For an entire function f and each $r > 0$, define $g(r) = \min\{|f(z)| : |z| = r\}$. Does this imply that g is monotone? **[1]**

   (e) Let f be an analytic function on a simply connected domain G such that $f(z) \neq 0$ for each $z \in G$. Is it possible to identify an analytic function h that satisfies $h'(z) = \frac{f'(z)}{f(z)}$ for all z in G? **[1]**

2. If f is an entire function that satisfies $|f(z)| \leq |z| + \frac{1}{\sqrt{|z|}}$ for each sufficiently large $|z|$, then show that f is injective. **[2]**

3. Let f be analytic on $B(a, R)$ and continuous on $\overline{B(a, R)}$. If there exists $c > 0$ such that $|f(z)| = c$ for all z on the circle $|z - a| = R$, then show that either f is constant on $\overline{B(a, R)}$ or f has at least one zero in $B(a, R)$. **[4]**

4. Determine all entire functions f that satisfy $f(0) \neq 0$ and the functional equation $f(3z) = (f(z))^3$ for all z in $\mathbb{C}$. **[5]**

5. Let f be analytic in an open set containing $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$. If $|f(z)| < 1$ for on the unit circle $|z| = 1$, then show that there exists a unique z with $|z| < 1$ such that $f(z) = z$. **[4]**

6. Does there exist a non-constant real-valued harmonic function u on $B(0, 1)$ such that $|u(z)| \leq u(z_o)$ for all z in $B(0, 1)$ and a fixed point $z_o \in B(0, 1)$? **[3]**

7. Find the number of zeros of $f(z) = z^7 + 4z^4 + z^3 + 1$ in the annulus $1 < |z| < 2$. **[3]**

8. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. For an analytic function $f : \mathbb{D} \to \mathbb{D}$, show that $\left|\frac{f(b) - f(a)}{1 - \bar{f}(b)f(a)}\right| \leq \left|\frac{b - a}{1 - \bar{b}a}\right|$ for every choice of a and b in $\mathbb{D}$. **[4]**

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9. Let f be an entire injective function that possesses a pole at infinity. Show that f can be expressed as $f(z) = az + b$. **5**

10. For $R > 0$, evaluate $\displaystyle\int_{|z|=R} \frac{dz}{z^2 \sin z}$. **5**

11. For $0 < a < 1$, show that $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = \frac{\pi}{\sin a\pi}$.
(Hint: Use a suitable rectangular contour for an appropriate complex function.) **5**

12. For $\alpha \in \mathbb{C} \setminus (\mathbb{N} \cup \{0\})$, let $f(z) = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \cdots$. Find the radius of convergence R of the power series. Show that $f'(z) = \dfrac{\alpha f(z)}{R+z}$ for $|z| < R$. Furthermore, deduce that $f(z) = (R+z)^\alpha$, whenever $|z| < R$. **5**

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