DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Guwahati

MA547: Complex Analysis — MidSem
Instructor: Rajesh Srivastava — February 24, 2025
Time duration: Two hours — Maximum Marks: 30

N.B. Answer without proper justification will attract zero mark.

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1. (a) Does there exist a non-constant entire function f such that $\bar{f}$ is also entire? $\boxed{1}$

   (b) Suppose u is a non-constant harmonic function on $\mathbb{R}^2$ and $T(x, y) = (2x, y + 1)$. Does it necessarily imply that $u \circ T$ is harmonic? $\boxed{1}$

   (c) Suppose R is the radius of convergence of power series $\sum a_n z^n$. Does it necessary imply that $f(z) = \sum a_n z^n$ is analytic on $|z| = R$? $\boxed{1}$

2. Is it true that $|\sin z| \leq |z|$ for each $z \in \mathbb{C}$? Show that there exists a small neighbourhood N of 0 and $k > 0$ such that $|\sin z| \leq k|z|$ for all $z \in N$. $\boxed{3}$

3. Let $D = \{z \in \mathbb{C} : |z| = R$ and $\text{Im}z \geq 0.\}$ For $R > 1$, show that the inequality $\left|\dfrac{e^{iz}}{z^2 + z + 1}\right| \leq \dfrac{1}{(R-1)^2}$ holds for each $z \in D$. $\boxed{4}$

4. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that the series $\sum \{1 - \cos(z^n)\}$ converges absolutely on $\mathbb{D}$. Does it converge uniformly on $\mathbb{D}$? $\boxed{3}$

5. Find all the possible analytic functions $f = u + iv : \mathbb{C} \to \mathbb{C}$ satisfying $u(x, y) - v(x, y) = x^2 - y^2$ on $\mathbb{C}$. Is it possible that v could be a constant? $\boxed{4}$

6. Find all possible analytic functions $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $z^2 f'(z) + f(z) = 0$ and $\lim\limits_{|z| \to \infty} f(z) = 1$. $\boxed{4}$

7. Find the radius of convergence of the power series $\sum 2^{n^2} z^{n(n+1)}$. $\boxed{2}$

8. Let α and $1+\alpha$ represent the reciprocal of radius of converge of power series $\sum a_n z^n$ and $\sum b_n z^n$ respectively. Prove that the radius of converge R of power series $\sum (a_n + b_n) z^n$ satisfies $\frac{1}{R} = 1 + \alpha$. $\boxed{4}$

9. Let $a, b \in \mathbb{C}$. Define $T(z) = az + b\bar{z}$. Show that $T : \mathbb{C} \to \mathbb{C}$ is bijective if and only if $|a| \neq |b|$. $\boxed{3}$

END