DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

Course: MA642: Real Analysis - I | **EndSem**
Instructor: Rajesh Srivastava | **Date:** November 18, 2025
Duration: 03:00 hours | **Maximum Marks:** 50

Note: Answers lacking rigorous justification will not be awarded marks.

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1. (a) Whether $f_n(x) = \sin^2(\frac{x}{n})$ forms an equicontinuous family in $C[0,1]$? **[1]**
   (b) Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $\{f(x - [x]) : x \in \mathbb{R}\}$ is unbounded? **[1]**
   (c) Whether $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $|f(x)| \leq \|x\|^2$ is necessarily continuously differentiable at $(0,0)$? **[1]**
   (d) Does there exists a discontinuous function on a metric space, whose graph is connected? **[1]**

2. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the liner transformation defined by $A(x,y) = (2x, 3y)$. Find the norm of A with respect to Euclidean norm. **[2]**

3. Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point. **[4]**

4. Let $X = [0,1] \cup [2,3]$. Show that there exists a non-constant uniformly continuous map $f : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = \emptyset$ and $f^{-1}((-\infty, 0)) \neq \emptyset \neq f^{-1}((0,\infty))$. **[4]**

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable such that $f_y(a,b) \neq 0$ for some $(a,b) \in \mathbb{R}^2$. Show that there exists a point $(c,d) \in \mathbb{R}^2$ such that $f(a,b) = f(c,d)$. **[5]**

6. Let $f : [0,1] \to \mathbb{R}$ satisfies IVP and set $\{x \in [0,1] : f(x) = t\}$ is closed for every $t \in \mathbb{R}$, then show that f is continuous. **[5]**

7. Let $f(x) = e^{-x^2}$. Show that there exists a polynomial p_k which converges uniformly to f on every compact subset of $\mathbb{R}$. **[5]**

8. Determine all functions $f \in C[0,1]$ such that $f(x) = \int_0^x (x-y) f(y)\, dy$. **[5]**

9. Show that there exists a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that the strict inequality $\|A^2\| < \|A\|^2$ holds. **[4]**

10. Show that equation $x^2 + yz - \cos(xz) = 0$ can be solved for x in some neighborhood of $(1,1,0)$. Whether it can be solved for y in a neighborhood of $(1,1,0)$? **[4]**

11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable with $f(0,0) = 0$. Find the condition under which $f(f(x,y), y) = 0$ can be solved for y in some neighborhood of $(0,0)$. **[3]**

12. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $f'(0) \neq 0$. Show that the function $g(x,y) = (f(x), xf(x) - y)$ is locally invertible in some neighborhood of $(0,0)$. Give an example of f (with justification) for which g is globally invertible. **[5]**

END