Advanced Hardy Spaces

M650 Lecture Notes, Jan–May 2022

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Contents

Notation index. The following conventions are used throughout these notes.

$\mathbb{D}$ the open unit disk in $\mathbb{C}$; $\overline{\mathbb{D}}$ its closure.

$\mathbb{T}$ the unit circle; m denotes the normalized Lebesgue measure on $\mathbb{T}$.

$\mathbb{C}_+$ the upper half-plane $\{z \in \mathbb{C} : \Im z > 0\}$.

$H^p(\mathbb{D})$ the Hardy space on $\mathbb{D}$ $(0 < p \leq \infty)$; similarly $H^p(\mathbb{C}_+)$.

P_r the Poisson kernel on $\mathbb{T}$ (disk model); P_y denotes the Poisson kernel on $\mathbb{R}$ (half-plane model).

B a (finite or infinite) Blaschke product; S_μ a singular inner function associated with a singular measure μ.

$\operatorname{inn}(f)$, $\operatorname{out}(f)$ the inner and outer factors of $f \in H^p$ (when defined).

$\mathcal{N}$, $\mathcal{N}^+$ the Nevanlinna and Smirnov classes.

Other symbols and standing assumptions are introduced locally at first use.

Preface

These notes were prepared for the course M650 (Jan–May 2022) at IIT Guwahati. Their aim is to introduce Hardy spaces as a meeting point of complex analysis and harmonic analysis, and to develop, in a self-contained way, the structural results that make the theory so useful.

Prerequisites. A reader should be comfortable with the basics of complex analysis (holomorphic functions, Cauchy's integral theorem and formula, power series) and real analysis (Lebesgue integration on $\mathbb{R}$, L^p spaces, and elementary Hilbert space theory). When we use a more advanced tool from functional analysis, it is stated explicitly and proved or referenced.

How the notes are organized. After preliminaries, we study shift-invariant subspaces of $L^2(\mathbb{T}, \mu)$ and the Beurling-type picture that underlies Hardy spaces. We then develop the canonical (inner–outer) factorization in $H^p(\mathbb{D})$, discuss Szegő-type theorems and the Nevanlinna/Smirnov classes, and finally transfer the theory to the upper half-plane $H^p(\mathbb{C}_+)$, where the Fourier transform and the Cauchy kernel provide a complementary viewpoint.

References and further reading. These notes are deliberately self-contained, but the subject has several excellent treatments with complementary styles. Standard references include Duren [13], Hoffman [10], Garnett [6], Koosis [4], and Nikolski [2, 3]. Where a proof is classical and would interrupt the main thread, we provide a short argument or a precise pointer.

How to use this edition. Each chapter begins with a short abstract together with learning objectives and a compact list of key ideas. When a section carries a roadmap, it indicates the logical order in which subsequent lemmas and propositions are used. The end-of-chapter exercise blocks are arranged as *guided examples*, *core exercises*, and occasional *extensions*; the comprehensive problem sets at the end are grouped into warm-up, core, and challenge tracks so that the notes can serve equally well for lectures, self-study, tutorials, or qualifying-exam preparation.

Syllabus

The course develops Hardy spaces as a meeting point of complex function theory, harmonic analysis, and invariant subspace methods. The emphasis is on structural theorems (invariant subspaces and canonical factorization) and on their applications to approximation, Fourier analysis, and weighted L^2-theory.

- **Invariant subspaces of $L^2(\mathbb{T}, \mu)$ (and more generally $L^2(\mu)$).** Doubly invariant subspaces. Simply invariant subspaces. Inner functions. Uniqueness theorem. Invariant subspaces of $L^2(\mathbb{T}, \mu)$.

- **Applications.** The problem of weighted polynomial approximation. A probabilistic interpretation. The inner–outer factorization. Arithmetic of inner functions. A characterization of outer functions. Fourier series. Szegő infimum and the F. & M. Riesz theorem.

- **H^p classes and canonical factorization (disk and circle).** Identifying $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$. Jensen's formula and Jensen's inequality. The boundary uniqueness theorem. Blaschke products. Non-tangential boundary limits. The Riesz–Smirnov canonical factorization. Approximation by inner functions and Blaschke products.

- **Szegő infimum and the Phragmén–Lindelöf principle.** Szegő infimum and weighted polynomial approximation. Recognize an outer function. Locally outer function. The Smirnov class N_+. A conformally invariant framework. The generalized Phragmén–Lindelöf principle.

- **Harmonic analysis on $L^2(\mathbb{T}; \mu)$.** Generalized Fourier series. Bases of exponentials in $L^2(\mathbb{T}; \mu)$. Harmonic conjugates. The Hilbert transform. The Helson–Szegő problem.

- **H^p spaces on the upper half-plane.** A unitary mapping from $L^p(\mathbb{T})$ to $L^p(\mathbb{R})$. Cauchy kernels and Fourier transform. The Hardy space $H^p_+ = H^p(\mathbb{C}_+)$. Canonical factorization and other relevant properties as compared to the disk. Invariant subspaces.

Guide to where each syllabus topic appears. The organization of the notes follows the syllabus closely, but the same theme often reappears from several viewpoints. As a quick map:

(I) *Invariant subspaces and inner functions:* Chapter 3 (doubly/simply invariant spaces, Beurling-type results).

(II) *Applications and weighted approximation:* Chapter 4 (weighted polynomial approximation, inner–outer factorization and its arithmetic).

(III) *H^p theory and canonical factorization in the disk:* Chapter 5 (boundary limits, Blaschke products, Riesz–Smirnov factorization, approximation issues).

Notation and conventions

- $\mathbb{D}$ denotes the unit disk, $\mathbb{T}$ the unit circle, and m the normalized arc-length measure on $\mathbb{T}$.
- For $1 \le p \le \infty$, $L^p(\mathbb{T})$ means $L^p(\mathbb{T}, m)$ unless another measure is specified.
- For $f \in L^1(\mathbb{T})$, the Fourier coefficients are

$$\widehat{f}(n) = \int_{\mathbb{T}} \overline{z}^n f(z)\, dm(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\, dt, \qquad n \in \mathbb{Z}.$$

- We use the standard Hardy space notation $H^p(\mathbb{D})$ (analytic functions on $\mathbb{D}$ with L^p boundary control) and $H^p(\mathbb{C}_+)$ for the upper half-plane model.

Chapter 1

Introduction

Hardy spaces form a bridge between complex analysis and harmonic analysis. They encode the boundary behaviour of holomorphic functions on the unit disk and the upper half-plane, and they interact in a precise way with Fourier series, singular integrals, and shift operators. These notes develop the basic structural results (Beurling-type theorems, inner–outer factorization, and canonical factorization) and then use them to study problems of approximation and invariant subspaces.

Learning objectives.

- Understand the definition of $H^p(\mathbb{D})$ through boundary values and Poisson extensions.
- See how Fourier analysis and the shift operator lead naturally to invariant subspaces and inner functions.
- Learn the role of inner–outer and canonical factorization in approximation and extremal problems.

Key ideas.

- Boundary behaviour is not auxiliary in Hardy-space theory: it is the main mechanism by which analytic information is encoded.
- Multiplication by z is simultaneously a geometric shift, a Fourier-frequency shift, and the basic operator behind invariant-subspace theory.
- Inner and outer factors separate "phase" and "size" in a way that later drives both approximation theorems and extremal formulas.

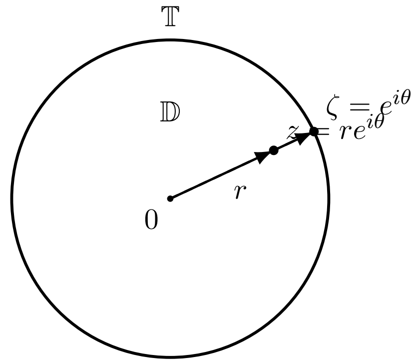

Figure 1.1: The unit disk and radial approach to the boundary (a basic geometric picture behind boundary values in $H^p(\mathbb{D})$).

Example 1.1 (The first inner functions)**.** For every integer $n \geq 0$, the monomial z^n is inner on $\mathbb{D}$. Consequently $z^n H^2$ is a closed shift-invariant subspace of H^2 and

$$H^2 \ominus z^n H^2 = \operatorname{span}\{1, z, \dots, z^{n-1}\}.$$

This simple model already foreshadows the role of inner functions in the structure theory of invariant subspaces.

Hardy introduced these spaces in 1915 in the context of power series and boundary growth. Over the subsequent decades, the subject was developed by many authors—notably the Riesz brothers, Szegő, Kolmogorov, Paley–Wiener, and later Beurling, Helson, and others—into a central toolkit of modern analysis. From the viewpoint of this course, the historical remark is mainly a guide: Hardy spaces are useful precisely because they package analytic information (holomorphy) together with quantitative boundary control (an L^p condition).

1.1 What is a Hardy space?

See Figure 1.1 for a schematic illustration.

For $0 < p \leq \infty$, the Hardy space $H^p(\mathbb{D})$ consists of holomorphic functions f on $\mathbb{D}$ whose boundary values are controlled in $L^p(\mathbb{T})$. One convenient definition is via radial means:

$$\|f\|_{H^p} := \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p}, \qquad (0 < p < \infty),$$

with the usual modification for $p = \infty$. A key theorem (Fatou) states that such f have non-tangential boundary limits $f^* \in L^p(\mathbb{T})$ and that f can be recovered from f^* by the

Poisson integral. Thus $H^p(\mathbb{D})$ may be viewed as a closed subspace of $L^p(\mathbb{T})$ consisting of functions whose negative Fourier coefficients vanish.

1.2 Invariant subspaces and inner functions

On $L^2(\mathbb{T})$, multiplication by z is an isometry (the *shift operator*). The closed subspaces invariant under this shift are governed by Beurling's theorem: every nontrivial closed subspace $E \subset H^2$ with $zE \subset E$ has the form $E = \Theta H^2$, where Θ is an *inner function* (analytic in $\mathbb{D}$ with unimodular boundary values a.e.). This result is one of the main structural pillars of the subject, and it explains why Hardy spaces are a natural playground for operator theory and functional analysis.

1.3 Organization of the notes

We begin with measure-theoretic preliminaries and the basic Fourier-analytic model of H^2. We then study shift-invariant subspaces of $L^2(\mathbb{T}, \mu)$ (Wiener, Wold–Kolmogorov, Helson) and derive first applications such as inner–outer factorization and Szegő-type extremal problems. Next we develop canonical factorization in $H^p(\mathbb{D})$, including Blaschke products, singular inner factors, and the Nevanlinna/Smirnov classes. Finally, we transfer the theory to the upper half-plane $H^p(\mathbb{C}_+)$, emphasizing the Fourier transform and the Cauchy kernel as complementary tools. Throughout, exercises and problem sets are included to help consolidate the ideas.

Chapter 2

Preliminaries and notation

This chapter fixes the analytic and measure-theoretic framework used throughout the notes. We recall the basic objects on the unit circle—normalized arc-length measure, Fourier coefficients, and the Hardy space boundary conventions—and we review complex Borel measures together with total variation. We also record the Riesz representation theorem and introduce the weak-$*$ topology on $\mathcal{M}(\mathbb{T})$, which will later underlie compactness and approximation arguments.

Learning objectives.

- Fix the measure-theoretic conventions and Fourier-analytic normalizations used throughout the notes.
- Review the Borel-measure formalism needed for analytic measures, singular parts, and duality arguments.
- Isolate the weak-$*$ compactness statements that later support approximation and extremal arguments.

Key ideas.

- Nearly every later theorem depends on keeping straight the distinction between boundary functions, measures, and Poisson extensions.
- Fourier coefficients are best viewed as a bridge between geometric objects on $\mathbb{T}$ and analytic objects on $\mathbb{D}$.

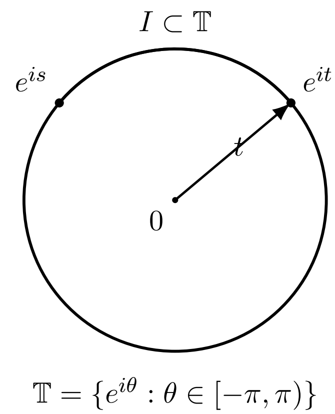

Figure 2.1: Parametrization of the unit circle and a typical arc $I \subset \mathbb{T}$ (useful for integrals with respect to normalized Lebesgue measure m).

- Weak-$*$ compactness enters not as abstract generality, but as a practical substitute for sequential compactness in spaces of measures.

Example 2.1 (A recurring kernel)**.** For $0 < r < 1$, the Poisson kernel

$$P_r(e^{it}) = \frac{1 - r^2}{|e^{it} - r|^2}$$

has Fourier coefficients $\widehat{P_r}(n) = r^{|n|}$. This identity is the template behind harmonic extension, radial approximation, and several summability arguments that appear later.

These notes use standard notation from complex analysis, measure theory, and basic functional analysis. For the reader's convenience we fix conventions that will be used throughout.

2.1 The unit circle and normalized Lebesgue measure

See Figure 2.1 for a schematic illustration.

We write

$$\mathbb{T} := \{z \in \mathbb{C} : \ |z| = 1\}, \qquad \mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}.$$

The parametrization $z = e^{it}$, $t \in [0, 2\pi)$ identifies $\mathbb{T}$ with the quotient group $\mathbb{R}/(2\pi\mathbb{Z})$ via the homomorphism $t \mapsto e^{it}$. Accordingly, any function $f : \mathbb{T} \to \mathbb{C}$ may be viewed as a 2π-periodic function on $\mathbb{R}$ by setting $f(t) := f(e^{it})$.

We denote by m the *normalized arc-length measure* on $\mathbb{T}$, i.e.

$$\int_{\mathbb{T}} f\, dm \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, dt, \qquad f \in L^1(\mathbb{T}, m).$$

With this normalization $m(\mathbb{T}) = 1$ and m is translation invariant:

$$\int_0^{2\pi} f(t - t_0)\, dt = \int_0^{2\pi} f(t)\, dt, \qquad t_0 \in [0, 2\pi).$$

2.2 Complex Borel measures and total variation

Let $\mathcal{B}(\mathbb{T})$ be the Borel σ-algebra of $\mathbb{T}$. A (finite) *complex Borel measure* on $\mathbb{T}$ is a countably additive map $\mu : \mathcal{B}(\mathbb{T}) \to \mathbb{C}$ with $\mu(\varnothing) = 0$ and

$$\mu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \sum_{j=1}^{\infty} \mu(B_j) \quad \text{for every disjoint family } \{B_j\}_{j\geq 1} \subset \mathcal{B}(\mathbb{T}),$$

where the series is absolutely convergent. The Banach space of all finite complex Borel measures on $\mathbb{T}$ will be denoted by $\mathcal{M}(\mathbb{T})$.

The *total variation* of $\mu \in \mathcal{M}(\mathbb{T})$ is the positive measure $|\mu|$ defined by

$$|\mu|(\mathbb{T}) = \sup\Big\{\sum_{j=1}^{\infty} |\mu(B_j)| : \ \{B_j\}_{j\geq 1} \text{ disjoint and } \bigcup_{j\geq 1} B_j = \mathbb{T}\Big\}.$$

The quantity $\|\mu\| := |\mu|(\mathbb{T})$ is the total variation norm, and $(\mathcal{M}(\mathbb{T}), \|\cdot\|)$ is a Banach space.

Exercise 2.2. Show that the definition of $|\mu|(\mathbb{T})$ is unchanged if the supremum is taken only over *finite* Borel partitions of $\mathbb{T}$.

Every $\mu \in \mathcal{M}(\mathbb{T})$ defines a bounded linear functional on $C(\mathbb{T})$ by

$$T_\mu(f) := \int_{\mathbb{T}} f\, d\mu, \qquad f \in C(\mathbb{T}),$$

and $\|T_\mu\| = \|\mu\|$. Conversely, every bounded linear functional on $C(\mathbb{T})$ arises this way.

Theorem 2.3 (Riesz representation theorem)**.** *For every bounded linear functional T on $C(\mathbb{T})$ there exists a unique $\mu \in \mathcal{M}(\mathbb{T})$ such that $T(f) = \int_{\mathbb{T}} f\, d\mu$ for all $f \in C(\mathbb{T})$. Equivalently, $\mathcal{M}(\mathbb{T}) \cong C(\mathbb{T})^*$ isometrically.*

2.3 The weak-* topology on $\mathcal{M}(\mathbb{T})$

Via Theorem 2.3 we identify $\mathcal{M}(\mathbb{T})$ with the dual space $C(\mathbb{T})^*$. The corresponding weak-* topology on $\mathcal{M}(\mathbb{T})$ will be denoted by w^*.

A typical w^*-neighborhood of $\mu_0 \in \mathcal{M}(\mathbb{T})$ is of the form

$$U(\mu_0; f_1, \ldots, f_N; \varepsilon) := \Big\{\mu \in \mathcal{M}(\mathbb{T}) : \ \Big|\langle \mu - \mu_0, f_k\rangle\Big| < \varepsilon,\ k = 1, \ldots, N\Big\},$$

where $f_1, \ldots, f_N \in C(\mathbb{T})$, $\varepsilon > 0$, and $\langle \mu, f\rangle := \int_{\mathbb{T}} f\, d\mu$.

We record a basic duality fact that will be used repeatedly.

Proposition 2.4. *Let E be a Banach space. A linear functional $\Phi : (E^*, w^*) \to \mathbb{C}$ is continuous if and only if there exists $x \in E$ such that $\Phi(f) = f(x)$ for all $f \in E^*$. Equivalently,*

$$(E^*, w^*)^* \cong E$$

via the canonical embedding $x \mapsto (f \mapsto f(x))$.

Proof. If $x \in E$, then $f \mapsto f(x)$ is w^*-continuous by definition of the weak-* topology.

Conversely, suppose Φ is w^*-continuous. Continuity at 0 means that there exist $x_1, \ldots, x_N \in E$ and $\varepsilon > 0$ such that

$$\Big(|f(x_1)| + \cdots + |f(x_N)| < \varepsilon\Big) \implies |\Phi(f)| < 1.$$

In particular, $\Phi(f) = 0$ whenever $f(x_k) = 0$ for all k, i.e. $\bigcap_{k=1}^N \ker(\mathrm{ev}_{x_k}) \subseteq \ker(\Phi)$. Therefore Φ factors through the finite-dimensional quotient $E^*/\bigcap_{k=1}^N \ker(\mathrm{ev}_{x_k})$, and hence can be written as a linear combination of the coordinate functionals $f \mapsto f(x_k)$. That is,

$$\Phi(f) = \sum_{k=1}^N c_k f(x_k) = f\Big(\sum_{k=1}^N c_k x_k\Big) \quad \text{for some } c_1, \ldots, c_N \in \mathbb{C}.$$

Setting $x := \sum_{k=1}^N c_k x_k$ gives the desired representation $\Phi(f) = f(x)$. □

Corollary 2.5. *The dual of $(\mathcal{M}(\mathbb{T}), w^*)$ is canonically isomorphic to $C(\mathbb{T})$.*

Chapter 3

Invariant subspaces of $L^2(\mathbb{T}, \mu)$

We study closed subspaces of $L^2(\mathbb{T}, \mu)$ that are invariant under multiplication by the coordinate function z. After establishing the basic Fourier-analytic formalism in the weighted setting, we prove a boundary uniqueness principle and develop Helson-type descriptions of simply invariant subspaces. These results provide the operator-theoretic backbone for the inner/outer machinery and for subsequent applications to approximation and cyclicity.

Learning objectives.

- Understand the distinction between reducing and simply invariant subspaces in the weighted L^2 setting.
- See how wandering vectors and cyclic subspaces lead to Beurling-type structure theorems.
- Connect uniqueness principles in Hardy spaces with invariant-subspace rigidity.

Key ideas.

- The operator M_z packages the geometry of Fourier modes into a single shift operator.
- Reducing subspaces are measure-theoretic in nature, whereas simply invariant subspaces carry genuinely analytic content.
- Boundary uniqueness is the mechanism that prevents nontrivial analytic functions from hiding on positive-measure boundary sets.

Example 3.1 (The model invariant subspace)**.** Inside $L^2(\mathbb{T},m)$, the Hardy space H^2 is invariant under multiplication by z, but it is not reducing because $\overline{z} \in H^{2\perp}$ while $z^{-1}H^2 \not\subset H^2$. Moreover

$$H^2 \ominus zH^2 = \mathbb{C}\cdot 1,$$

so the shift $M_z|_{H^2}$ is a unilateral shift of multiplicity one.

In this chapter we study shift-invariant subspaces of square-integrable functions on $\mathbb{T}$. Let

$$L^2(\mathbb{T},\mu) = \left\{ f:\mathbb{T}\to\mathbb{C}:\ f \text{ is measurable and } \|f\|_2^2 = \int_{\mathbb{T}} |f|^2\, d\mu < \infty \right\},$$

where μ is a finite positive Borel measure on $\mathbb{T}$.

For $f \in L^1(\mathbb{T},m)$, we define the Fourier coefficients of f by

$$\widehat{f}(n) = \int_{\mathbb{T}} \overline{z}^n f(z)\, dm(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\, dt, \qquad n\in\mathbb{Z}.$$

The corresponding Fourier series is written as $f \sim \sum_{n=-\infty}^{\infty} e^{int}\hat{f}(n)$. Consider the operator S on $L^2(\mathbb{T},m)$ defined by

$$S(f)(z) = zf(z), \tag{3.0.1}$$

for $z\in\mathbb{T}$. Then $\widehat{(Sf)}(n) = \hat{f}(n-1)$, so the action of S shifts the Fourier coefficients one step to the right. For this reason S is called the shift operator. The basic question is the following.

Question 3.2. What are the shift-invariant subspaces E of $L^2(\mathbb{T},\mu)$?

In other words, when does $zE \subseteq E$ hold? We write $\operatorname{clos} E$ for the closure of E and $\bar{E}$ for the complex conjugate of E. Unless stated otherwise, subspaces are always assumed to be closed.

Example 3.3. For $f\in L^2(\mu)$, the space

$$E_f = \overline{\operatorname{span}}\{z^n f : n\geq 0\}$$

is shift-invariant.

A natural question is to determine when $E_f = L^2(\mu)$; in that case, f is called a *cyclic vector*. More generally, one would like to understand those $f\in L^2(\mu)$ for which $zE_f = E_f$.

Let E be a closed subspace of L^2. The two basic cases are as follows.

We say that E is **simply invariant** (or **1-invariant**) if $zE \subset E$ and $zE \neq E$. We say that E is **doubly invariant** (or **2-invariant**) if $zE = E$. Note that $zE = E$ if and only if $\bar{z}E = E$ (since $z\bar{z} = |z|^2 = 1$). Equivalently, $zE \subseteq E$ and $\bar{z}E \subseteq E$, so E is also called a *reducing subspace*.

For a measurable set $\sigma \subset \mathbb{T}$, the space $E_\sigma = \chi_\sigma L^2(\mu) = \{\chi_\sigma f : f \in L^2(\mu)\} = \{f \in L^2(\mu) : f = 0 \ \mu\text{-a.e. on } \mathbb{T} \setminus \sigma\}$ satisfies $zE_\sigma = E_\sigma$.

Question 3.4. Does every reducing subspace look like E_σ?

Theorem 3.5. (Norbert Wiener) *Let $E \subset L^2(\mathbb{T}, \mu)$. Then $zE = E$ if and only if there exists a unique (up to set of measure zero) measurable set $\sigma \subset \mathbb{T}$ such that $E = \chi_\sigma L^2(\mu)$.*

Proof. Assume that $zE = E$, and let P_E denote the orthogonal projection of $L^2(\mathbb{T}, \mu)$ onto E. Set $\chi := P_E 1$. Then $\chi \in E$ and $1 - \chi \in E^\perp$.

Since multiplication by z is unitary on $L^2(\mu)$ and $zE = E$, we have $z^n\chi \in E$ and $z^n(1-\chi) \in E^\perp$ for every $n \in \mathbb{Z}$. Hence

$$0 = \langle z^n\chi, \, 1 - \chi\rangle_{L^2(\mu)} = \int_{\mathbb{T}} z^n \chi \, \overline{(1-\chi)} \, d\mu = \int_{\mathbb{T}} \overline{z}^{\,n} \, \chi(1-\overline{\chi}) \, d\mu, \qquad n \in \mathbb{Z}.$$

Let $d\nu := \chi(1-\overline{\chi}) \, d\mu$. The display shows that all Fourier coefficients of the complex measure ν vanish, so $\nu = 0$ by Lemma 3.10(i). Consequently,

$$\chi(1-\overline{\chi}) = 0 \quad \mu\text{-a.e.},$$

which implies $\chi = |\chi|^2$ μ-a.e. and therefore $\chi = \chi_\sigma$ for some measurable $\sigma \subset \mathbb{T}$ (unique up to μ-null sets).

Since $\chi \in E$ and $zE = E$, we have $z^n\chi \in E$ for all $n \in \mathbb{Z}$. Thus $\chi \, \mathbb{P} \subset E$, where $\mathbb{P}$ is the space of trigonometric polynomials. As $\mathbb{P}$ is dense in $L^2(\mu)$, it follows that $\chi L^2(\mu) \subset E$.

Conversely, let $f \in E$. For each $n \in \mathbb{Z}$ we have $z^n f \in E$ and $1 - \chi \in E^\perp$, hence

$$\int_{\mathbb{T}} z^n f \, \overline{(1-\chi)} \, d\mu = 0, \qquad n \in \mathbb{Z}.$$

Equivalently, all Fourier coefficients of the complex measure $f(1-\overline{\chi}) \, d\mu$ vanish. By Lemma 3.10(i) we obtain $f(1-\overline{\chi}) = 0$ μ-a.e., so $f = \chi f \in \chi L^2(\mu)$. Therefore $E \subset \chi L^2(\mu)$.

We conclude that $E = \chi_\sigma L^2(\mu)$. The converse implication is immediate: if $E = \chi_\sigma L^2(\mu)$ then $zE = E$ because $|z| = 1$ on $\mathbb{T}$. □

3.1 Simply invariant subspaces of $L^2(\mu)$

Let $\mathcal{B} = \{z^n\}_{n\in\mathbb{Z}}$. Notice that the Fourier series of $f \in L^2(\mathbb{T}, m)$ with respect to the orthonormal basis $\mathcal{B}$ is $f \sim \sum \hat{f}(n) z^n$, where $\hat{f}(n) = \int\limits_{\mathbb{T}} f \bar{z}^n dm$. This implies that $L^2(\mathbb{T}, m)$ can be identified with $l^2(\mathbb{Z})$. Since $\widehat{(z^k f)}(n) = \hat{f}(n-k)$, multiplication operator $f \mapsto zf$ acts as a right-shift operator on $l^2(\mathbb{Z})$. Hence it is legitimate to consider the space

$$H^2 = \overline{\text{span}}\{z^n : n \geq 0\} = \{f \in L^2(m) : \hat{f}(n) = 0, n < 0\},$$

known as **Hardy space**. The space H^2 is a simply invariant subspace of $L^2(m)$, and plays a prominent role in complex and harmonic analysis H^2.

The following theorem says that all the simply invariant subspaces have a somewhat similar structure.

Theorem 3.6. (A. Beurling, H. Helson) *Let E be a closed subspace of $L^2(\mathbb{T})$ and $zE \subset E$, $zE \neq E$. Then there exists a unique θ (up to constant of modulus 1) with $|\theta| = 1$ a.e. m on $\mathbb{T}$ such that $E = \theta H^2$.*

Notice that $f \mapsto \theta f$ is an isometry on $L^2(m)$, and hence θH^2 is closed.

Proof. Set $D := E \ominus zE$. Since E is simply invariant, $D \neq \{0\}$. Choose $\theta \in D$ with $\|\theta\|_{L^2(m)} = 1$. Then $\theta \perp z^n\theta$ for every $n \geq 1$, and therefore

$$0 = \langle z^n\theta, \theta\rangle_{L^2(m)} = \int_{\mathbb{T}} |\theta|^2 z^n \, dm, \qquad n \geq 1.$$

Taking complex conjugates gives $\int_{\mathbb{T}} |\theta|^2 \bar{z}^n \, dm = 0$ for all $n \geq 1$. Hence all nonzero Fourier coefficients of $|\theta|^2$ vanish, so $|\theta|^2$ is constant a.e. on $\mathbb{T}$. Since $\|\theta\|_2 = 1$, this constant equals 1, and thus $|\theta| = 1$ a.e. on $\mathbb{T}$.

Because $\theta \in E$ and $zE \subset E$, we have $z^n\theta \in E$ for every $n \geq 0$. Hence $\theta\,\mathbb{P}_+ \subset E$, where $\mathbb{P}_+ = \text{span}\{z^n : n \geq 0\}$, and therefore $\theta H^2 = \overline{\theta\,\mathbb{P}_+} \subset E$.

To prove the reverse inclusion, let $f \in E$ with $f \perp \theta H^2$. Then

$$\int_{\mathbb{T}} f\,\overline{\theta}\,\overline{z}^n \, dm = 0, \qquad n \geq 0.$$

Moreover, for each $n \geq 1$ we have $z^n f \in zE$ and $\theta \perp zE$, hence

$$\int_{\mathbb{T}} z^n f\,\overline{\theta}\, dm = 0, \qquad n \geq 1.$$

Together, these relations say that all Fourier coefficients of $f\overline{\theta}$ vanish. Thus $f\overline{\theta} = 0$ a.e. on

$\mathbb{T}$, and since $|\theta| = 1$ a.e. we obtain $f = 0$. Therefore $E = \theta H^2$.

Uniqueness. If $\theta_1 H^2 = \theta_2 H^2$ and $|\theta_1| = |\theta_2| = 1$ a.e. on $\mathbb{T}$, then $\theta_1\overline{\theta_2}\, H^2 = H^2$, so $\theta_1\overline{\theta_2} \in H^2$. By symmetry, $\overline{\theta_1}\theta_2 \in H^2$ as well, hence $\theta_1\overline{\theta_2} \in H^2 \cap \overline{H^2}$. Since $H^2 \cap \overline{H^2}$ consists of constants, we have $\theta_1 = \lambda\theta_2$ for some $\lambda \in \mathbb{T}$. □

Corollary 3.7. (Beurling theorem) *Let $E \neq \{0\}, E \subset H^2$ and $zE \subset E$. Then there exists $\theta \in H^2$ with $|\theta| = 1$ a.e. on $\mathbb{T}$ such that $E = \theta H^2$.*

Proof. If $E \subset H^2$ is nonzero and satisfies $zE \subset E$, then either $zE \subsetneq E$ or $zE = E$. We claim that the second alternative is impossible.

Indeed, if $zE = E$, then multiplication by z is a surjective isometry on E and hence its inverse m_z^{-1} is given by multiplication by $\overline{z}$; thus $\overline{z}E \subset E$. Pick $0 \neq f \in E$ and let

$$n_0 := \min\{n \geq 0 : \widehat{f}(n) \neq 0\}.$$

Then $g := \overline{z}^{n_0+1} f \in E$ under the assumption $\overline{z}E \subset E$. However, $\widehat{g}(-1) = \widehat{f}(n_0) \neq 0$, so $g \notin H^2$, contradicting $E \subset H^2$. Therefore $zE \subsetneq E$, i.e. E is simply invariant, and Theorem 3.6 yields $E = \theta H^2$ for an inner function θ. Since $E \subset H^2$, necessarily $\theta \in H^2$. □

Definition 3.8. A function $\theta \in H^2$, with $|\theta| = 1$ a.e. is called **inner** function.

3.2 Uniqueness theorem in H^2

Theorem 3.9. *If $f \in H^2$ and $f = 0$ on a set of positive measure, then $f = 0$ a.e. on $\mathbb{T}$.*

Proof. Let $\sigma := \{z \in \mathbb{T} : f(z) = 0\}$. Assume $m(\sigma) > 0$. If $f \not\equiv 0$, consider the shift-invariant subspace

$$E_f := \overline{\operatorname{span}}\{z^n f : n \geq 0\} \subset H^2.$$

By Corollary 3.7, there exists an inner function θ such that $E_f = \theta H^2$.

We claim that every $g \in E_f$ satisfies $g = 0$ a.e. on σ. Indeed, choose polynomials $p_k \in \mathbb{P}_+$ with $p_k f \to g$ in $L^2(m)$. Since $p_k f = 0$ on σ,

$$\int_\sigma |g|^2\, dm = \int_\sigma |g - p_k f|^2\, dm \leq \|g - p_k f\|^2_{L^2(m)} \longrightarrow 0.$$

Thus $g|_\sigma = 0$ a.e. on σ. In particular, $\theta \in E_f$ implies $\theta = 0$ a.e. on σ. This contradicts the fact that θ is inner, hence $|\theta| = 1$ a.e. on $\mathbb{T}$. Therefore $f \equiv 0$. □

3.3 Invariant subspaces of $L^2(\mu)$

(Absolutely continuous and singular subspaces)

Let μ be a finite Borel measure on $\mathbb{T}$, and $E \subset L^2(\mu)$ with $zE \subset E$. We consider invariant subspaces of $L^2(\mu)$ which are based on Lebesgue decomposition of μ. A measure ν is called **absolutely continuous** with respect to m if $m(B) = 0$ implies $\nu(B) = 0$, where $B \in \mathcal{B}$ and we write $\nu \ll m$. By Radon-Nikodym theorem, there exists a positive integrable function w such that $d\nu = wdm$. That is,

$$\int_{\mathbb{T}} f d\nu = \int_{\mathbb{T}} f w dm$$

for each Borel measurable function f on $\mathbb{T}$.

A measure ν is called **singular** with respect to m if it is concentrated on a set C of Lebesgue measure zero. That is, $\nu \perp m$ if $\nu(B) = m(B \cap C)$ for every $B \in \mathcal{B}(\mathbb{T})$. Let μ be a finite and positive Borel measure on $\mathbb{T}$, then by Lebesgue decomposition,

$$\mu = \mu_a + \mu_s, \text{ where } \mu_a \ll m \text{ and } \mu_s \perp m.$$

So, if $f \in L^2(\mu)$, then

$$\int_{\mathbb{T}} |f|^2 d\mu = \int_{\mathbb{T}} |f|^2 d\mu_a + \int_{\mathbb{T}} |f|^2 d\mu_s$$

By this, we can construct an orthogonal decomposition of f. Let σ be the concentration set for μ_s. Then

$$L^2(\mu_s) \subset L^2(\mu) \text{ and } L^2(\mu_a) \subset L^2(\mu) \text{ and } L^2(\mu_s) \perp L^2(\mu_a). \tag{3.3.1}$$

Now, $f = f\chi_{\mathbb{T}\setminus\sigma} + f\chi_\sigma = f_a + f_s$. This means

$$L^2(\mu) = L^2(\mu_a) \oplus L^2(\mu_s). \tag{3.3.2}$$

The subspaces $L^2(\mu_a)$ and $L^2(\mu_s)$ are invariant subspaces and are known as absolutely continuous and singular spaces, respectively.

We need the following results in order to prove the main result about invariant subspaces of $L^2(\mu)$.

Lemma 3.10. *Let μ be a finite complex Borel measure on $\mathbb{T}$.*

(i) If $(\widehat{d\mu})(n) = \int_{\mathbb{T}} e^{-int} d\mu(t) = 0$ for all $n \in \mathbb{Z}$, then $\mu = 0$.

(ii) If $\widehat{(d\mu)}(n) = 0$ *for all* $n \in \mathbb{Z} \setminus \{0\}$, *then* $d\mu = cdm$.

Proof. (i) Let $f \in C^2(\mathbb{T})$, then f is Borel measurable and we have

$$\begin{aligned} T_\mu(f) &= \int_{\mathbb{T}} f(t)d\mu(t) \\ &= \int_{\mathbb{T}} \Big(\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}\Big)d\mu(t) \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n)\int_{\mathbb{T}} e^{int}d\mu(t) \text{ (by Fubini's Theorem)} \\ &= 0 \text{ (by assumption).} \end{aligned}$$

Hence $T_\mu(f) = 0$ for all $f \in C^2(\mathbb{T})$. Since $C^2(\mathbb{T})$ is dense in $C(\mathbb{T})$, by Theorem 2.3, we get $\mu = 0$.
(ii) From the given condition and similar to the proof of case (i), we can write

$$\int_{\mathbb{T}} f(t)d\mu(t) = \hat{f}(0)\int_{\mathbb{T}} d\mu = \mu(\mathbb{T})\int_{\mathbb{T}} f(t)dt.$$

Thus $d\mu = \mu(\mathbb{T})dm$, where $dm = dt$. □

Let $T : H \to H$ be an isometry (or $T \in \text{iso}(H)$) on the Hilbert space H. A subspace D of H is called **wandering** if $T^mD \perp T^nD$ for $m \neq n\, (m, n \geq 0)$.

Lemma 3.11. (H. Wold, A. Kolmogorov) *Suppose* $T \in iso(H)$ *and* $TE \subset E$. *Let* $D = E \ominus TE$. *Then* D *is a wandering subspace of* T, *and* $E = \Big(\sum_{n\geq 0} \oplus T^nD\Big) \oplus \Big(\bigcap_{n\geq 0} T^nE\Big) = E_0 \oplus E_\infty$, *where* $T|_{E_\infty}$ *is unitary, and* $T|_{E_0}$ *is completely non-unitary (i.e. if* $E' \subset E_0$ *and* $TE' \subset E'$ *implies* $T|_{E'}$ *is not unitary).*

Theorem 3.12. (H. Helson 1964) *Let* $d\mu = wdm + d\mu_s$ *be the Lebesgue decomposition of a positive finite Borel measure* μ *and let* $E \subset L^2(\mu)$ *be simply invariant. Then there exists* $\sigma \subseteq \mathbb{T}$ *with* $m(\sigma) = 0$ *and a measurable function* θ *such that*

$$E = E_0 \oplus E_\infty = \theta H^2 \oplus \chi_\sigma L^2(\mu_s), \text{ where}$$

$$\theta H^2 \subset L^2(\mu_a),\ \chi_\sigma L^2(\mu_s) \subset L^2(\mu_s) \text{ and}$$

$$|\theta|^2 w \equiv 1. \tag{3.3.3}$$

Conversely, if σ *is measurable and* θ *verified (3.3.3), then* $\theta H^2 \oplus \chi_\sigma L^2(\mu_s)$ *is simply invariant.*

Proof. Let M_z denote multiplication by z on $L^2(\mathbb{T},\mu)$. Since $|z|=1$ on $\mathbb{T}$, the operator M_z is unitary. Assume that $E\subset L^2(\mu)$ is simply invariant, and set

$$D := E\ominus zE, \qquad E_\infty := \bigcap_{n\ge 0} z^nE, \qquad E_0 := \overline{\bigoplus_{n\ge 0} z^nD}.$$

By the Wold–Kolmogorov decomposition (Lemma 3.11) applied to $M_z|_E$, we have $E=E_0\oplus E_\infty$ and M_z is unitary on E_∞.

Step 1: construction of θ and the identity $|\theta|^2w\equiv 1$. Choose $\theta\in D$ with $\|\theta\|_{L^2(\mu)}=1$. Then $\theta\perp z^n\theta$ for all $n\ge 1$, so

$$\int_{\mathbb{T}} |\theta|^2 z^n\,d\mu = 0, \qquad n\ge 1,$$

and taking complex conjugates yields the same for $\overline{z}^n$, $n\ge 1$. Thus the complex measure $|\theta|^2\,d\mu$ has vanishing Fourier coefficients at every nonzero index. By Lemma 3.10(ii) we obtain $|\theta|^2\,d\mu = c\,dm$ for some $c\ge 0$. Since $\|\theta\|^2_{L^2(\mu)} = \int_{\mathbb{T}}|\theta|^2\,d\mu = 1$ and $m(\mathbb{T})=1$, we have $c=1$. Therefore

$$dm = |\theta|^2\,d\mu = |\theta|^2w\,dm + |\theta|^2\,d\mu_s. \tag{3.3.4}$$

Comparing the Lebesgue decompositions in (3.3.4) shows that $|\theta|^2\,d\mu_s=0$, hence $\theta=0$ μ_s-a.e., and that $|\theta|^2w=1$ m-a.e.

In particular, $w>0$ a.e. m, and multiplication by θ defines an isometry $M_\theta : L^2(\mathbb{T},dm)\to L^2(\mathbb{T},\mu_a)$.

Step 2: description of E_∞. Since M_z is unitary on E_∞, we have $zE_\infty=E_\infty$. By Wiener's theorem (Theorem 3.5), there exists a measurable $\sigma\subset\mathbb{T}$ such that $E_\infty=\chi_\sigma L^2(\mu)$. Because $\theta\in D\subset E_0\perp E_\infty$, we have $\theta\perp\chi_\sigma L^2(\mu)$, hence $\theta=0$ μ-a.e. on σ. Since $|\theta|^2w=1$ a.e. m, this forces $m(\sigma)=0$. Consequently, $\chi_\sigma L^2(\mu)=\chi_\sigma L^2(\mu_s)\subset L^2(\mu_s)$.

Step 3: identification of E_0 with θH^2. We already know $E_0=\overline{\bigoplus_{n\ge0} z^nD}\subset L^2(\mu_a)$ because $D\perp E_\infty$ and $E_\infty\supset L^2(\mu_s)$. We claim that

$$E_0 = \overline{\operatorname{span}}\{z^n\theta : n\ge 0\}.$$

Indeed, suppose $f\in E_0$ is orthogonal to $z^n\theta$ for all $n\ge 0$. Since $f\in E$ we also have $z^nf\in zE$ for every $n\ge 1$, and $\theta\perp zE$ implies $\langle z^nf,\theta\rangle_{L^2(\mu)}=0$ for all $n\ge 1$. Thus all Fourier coefficients of the complex measure $f\overline{\theta}\,d\mu$ vanish, and by Lemma 3.10(i) we obtain $f\overline{\theta}=0$ μ-a.e. Because $\overline{\theta}\ne 0$ a.e. m and $f\in L^2(\mu_a)$, it follows that $f=0$. This proves the claim.

Next write $f \in E_0$ as $f = \sum_{n\geq 0} a_n z^n \theta$ with $(a_n) \in \ell^2$. Since $|\theta|^2 w \equiv 1$, the family $\{z^n\theta\}_{n\geq 0}$ is orthonormal in $L^2(\mu_a)$, so the map

$$H^2 \ni \sum_{n\geq 0} a_n z^n \longmapsto \theta \sum_{n\geq 0} a_n z^n \in L^2(\mu_a)$$

is an isometry and has range precisely E_0. Hence $E_0 = \theta H^2$.

Combining Steps 2 and 3 we obtain

$$E = E_0 \oplus E_\infty = \theta H^2 \oplus \chi_\sigma L^2(\mu_s),$$

with $m(\sigma) = 0$ and $|\theta|^2 w \equiv 1$.

Conversely. If σ is measurable and θ is measurable with $|\theta|^2 w \equiv 1$ a.e. m, then

$\theta H^2 \subset L^2(\mu_a)$ is simply invariant, and $\chi_\sigma L^2(\mu_s) \subset L^2(\mu_s)$ is doubly invariant. Their orthogonal direct sum is therefore simply invariant. □

Chapter 4

First Applications

We develop the first consequences of Helson's theorem and the invariant-subspace viewpoint. Topics include reducing subspaces, the weighted polynomial approximation problem, and a systematic treatment of inner–outer factorization and the algebra of inner functions. We conclude with the integral maximum principle characterization of outer functions and a first encounter with Szegő-type infimum problems.

Learning objectives.

- Use invariant-subspace methods to derive concrete approximation and cyclicity statements.
- Understand the first version of inner–outer factorization and its algebraic consequences.
- Interpret the Szegő infimum both analytically and geometrically.

Key ideas.

- Helson's theorem becomes most useful when translated into density and distance questions.
- Outer functions are exactly the analytic functions whose cyclic span is as large as possible.
- Weighted approximation problems are rarely isolated phenomena; they encode the same structure as extremal formulas and prediction problems.

Example 4.1 (A basic outer polynomial)**.** If $|a| < 1$, then $1 - \overline{a}z$ has no zeros in $\mathbb{D}$ and is therefore outer in H^2. By contrast, the normalized factor

$$b_a(z) = \frac{a - z}{1 - \overline{a}z}$$

is inner. This elementary dichotomy already illustrates the complementary roles of zero sets and boundary modulus.

We have already seen, via Helson's theorem, that simply invariant subspaces of $L^2(\mu)$ are in one-to-one correspondence with measurable unimodular functions. This point of view allows Hilbert-space geometry and operator theory to be brought to bear on $L^2(\mu)$, and conversely. In this chapter we study several first applications: inner–outer factorization for Hardy-space functions, the Szegő infimum, and the F. & M. Riesz theorem for analytic measures. A recurring theme is the weighted approximation problem: for which positive measures μ on $\mathbb{T}$ is the "analytic half"

$$\mathbb{P}_+ = \text{span}\{z^n : n \geq 0\}$$

dense in $L^2(\mathbb{T}, \mu)$?

4.1 Some consequences of Helson's theorem

Section roadmap.

- First isolate the absolutely continuous and singular pieces of an invariant subspace.
- Then translate Helson's theorem into concrete density and approximation statements.
- Finally interpret these statements in terms of cyclicity and weighted polynomial approximation.

Let μ be a positive Borel measure on $\mathbb{T}$ with $d\mu = wdm + d\mu_s$. Notice that if $zE \subset E \subset L^2(\mu)$, then $E = E_a \oplus E_s$, where $zE_a \subset E_a \subset L^2(\mu_a)$, because $E = \theta H^2 \oplus \chi_\sigma L^2(\mu_s)$, where $\theta H^2 \subset L^2(\mu_a)$ and $\chi_\sigma L^2(\mu_s) \subset L^2(\mu_s)$.

(a) If $\mu = \mu_s$, then $zE \subset E \subset L^2(\mu_s)$, implies $zE = E$, because, by Helson's theorem 3.12, we already have $E = \chi_\sigma L^2(\mu_s)$, which is 2-invariant.

(b) Show that for $d\mu = d\mu_a = w\,dm$, the following are equivalent:

(i) There exists E such that $zE \subsetneq E \subset L^2(\mu_a)$.

(ii) There exists θ such that $|\theta|^2 w = 1$ a.e. m.

(iii) $w > 0$ almost everywhere m.

(iv) m is absolutely continuous with respect to μ_a.

(c) If $d\mu = d\mu_a = w\,dm$ and $zE \subsetneq E \subset L^2(\mu_a)$, then $E = \theta H^2$ with $|\theta|^2 w \equiv 1$ a.e. m.

4.2 Reducing subspaces

Let $f \in L^2(\mu)$ and $d\mu = wdm + d\mu_s$. We look for sufficient conditions that ensure that E_f is reducing. If there exists measurable set $e \subset \mathbb{T}$ such that $m(e) > 0$ and $f|_e = 0$. Then E_f is a reducing subspace, and there exists $\sigma \subset \mathbb{T} \smallsetminus e$ such that $E_f = \chi_\sigma L^2(\mu)$. In fact, $\sigma = \{z \in \mathbb{T} : f(z) \neq 0\}$. On the contrary, suppose $zE_f \subsetneq E_f$. Then by Theorem 3.12 we get $E_f = \theta H^2 \oplus \chi L^2(\mu_s)$, and hence $f \in E_f$ implies $f = f_a + f_s$, where $f_a = \theta h$, $h \neq 0$ a.e. m (by Theorem 3.9, since $h \in H^2$). This implies $f_a \neq 0$ a.e. m, which is impossible because $f|_e = 0$ and $m(e) > 0$ implies $f_a|_e = 0$ with $m(e) > 0$. Thus, $E_f = zE_f = \chi_\sigma L^2(\mu)$ for $\sigma \subset \mathbb{T}$ (by Wiener theorem). Notice that $E_f = \overline{\text{span}}\{z^n \chi_{\mathbb{T}\smallsetminus e} f : n \geq 0\} = \chi_{\mathbb{T}\smallsetminus e} E_f = \chi_\sigma L^2(\mu)$ and $1 \in L^2(\mu)$, implies $\sigma \subset \mathbb{T} \setminus e$. Indeed $\sigma = \{z \in \mathbb{T} : f(z) \neq 0\}$, which is defined up to a set of μ measure zero.

4.3 The problem of weighted polynomial approximation

We know that the space of trigonometric polynomials $\mathbb{P} = \text{span}\{z^n : n \in \mathbb{Z}\}$ is dense in $L^p(\mu)$ for every positive and finite measure μ and $1 \leq p < \infty$. Let $\mathbb{P}_+ = \text{span}\{z^n : n \geq 0\}$. One of the main problems is describing the closure of $\mathbb{P}_+$ in $L^2(\mu)$. Denote $H^2(\mu) = \text{clos}\, \mathbb{P}_+|_{L^2(\mu)}$. The most important part of this problem is to distinguish between the completeness case $H^2(\mu) = L^2(\mu)$, from the incompleteness case $H^2(\mu) \subsetneq L^2(\mu)$.

Corollary 4.2. $H^2(\mu) = H^2(\mu_a) \oplus L^2(\mu_s)$.

Proof. $H^2(\mu) = \overline{\text{span}}\{z^n : n \geq 0\}$. By Helson decomposition $H^2(\mu) = E_a \oplus E_s$ with $E_a \subset L^2(\mu_a)$ and $E_s \subset L^2(\mu_s)$. Since we know that $zE_s = E_s$, by Wiener theorem,

$E_s = \chi_\sigma L^2(\mu_s)$ with $m(\sigma) = 0$. Since $1 \in H^2(\mu)$, we have $1 = 1_a + 1_s$ with $1_s \neq 0$ a.e. μ_s. But $1_s \in E_s = \chi_\sigma L^2(\mu_s)$ implies $\chi_\sigma L^2(\mu_s) = L^2(\mu_s)$, i.e., $E_s = L^2(\mu_s)$.

Further, $(\mathbb{P}_+)_a \subset E_a$ implies $\operatorname{clos}(\mathbb{P}_+)_a = H^2(\mu_a) \subseteq E_a$. But, for $f \in E_a \subset H^2(\mu)$ implies there exists $p_n \in \mathbb{P}_+$ such that $\|f - p_n\|_{L^2(\mu)} \to 0$. Since $\|f - p_n\|^2_{L^2(\mu)} = \|f - p_n\|^2_{L^2(\mu_a)} + \|f - p_n\|^2_{L^2(\mu_s)} = \|f - p_n\|^2_{L^2(\mu_a)} + \|p_n\|^2_{L^2(\mu_s)}$ (since $f = 0$ μ_s-a.e.) and $\|f - p_n\|^2_{L^2(\mu_a)} \leq \|f - p_n\|^2_{L^2(\mu_a)} + \|p_n\|^2_{L^2(\mu_s)} = \|f - p_n\|^2_{L^2(\mu)} \to 0$ we get $f \in H^2(\mu_a)$. □

Remark 4.3. For $H^2(\mu_a)$, the closure of $\mathbb{P}_+$ in $L^2(\mu_a)$ has exactly two possibilities:

(i) $zH^2(\mu_a) = H^2(\mu_a)$, in which case Wiener's theorem gives $H^2(\mu_a) = \chi_\sigma L^2(\mu_a) = L^2(\mu_a)$. Since $1_a \in H^2(\mu_a)$, this forces $\sigma = \mathbb{T}$ up to an m-null set.

(ii) $zH^2(\mu_a) \subsetneq H^2(\mu_a) \subset L^2(\mu_a)$, in which case $H^2(\mu_a) = \theta H^2$ for some θ satisfying $|\theta|^2 w \equiv 1$.

The following results help to distinguish the above two cases.

Lemma 4.4. *$H^2(\mu)$ is reducing (and hence $H^2(\mu) = L^2(\mu)$) if and only if $\bar{z} \in H^2(\mu)$.*

Proof. If $H^2(\mu)$ is reducing, then $\bar{z} \in H^2(\mu)$ is trivial. Suppose $\bar{z} \in H^2(\mu)$, then exists $p_n \in \mathbb{P}_+$ such that $\|\bar{z} - p_n\|_{L^2(\mu)} \to 0$. Let $q \in \mathbb{P}_+$. Then

$$\int_{\mathbb{T}} |\bar{z}q - qp_n|^2 d\mu \leq \|q\|^2_\infty \int_{\mathbb{T}} |\bar{z} - p_n|^2 \to 0 \text{ as } n \to \infty.$$

This implies $\bar{z}\,\mathbb{P}_+ \subset H^2(\mu)$, or $\mathbb{P}_+ \subset zH^2(\mu)$ (closed). Hence $H^2(\mu) \subseteq zH^2(\mu)$, i.e. $\bar{z}H^2(\mu) \subseteq H^2(\mu)$. But $zH^2(\mu) \subset H^2(\mu)$ implies $zH^2(\mu) = H^2(\mu)$. Next, it follows from Wiener theorem and theorem 3.9 that $H^2(\mu) = \chi_\sigma L^2(\mu) = L^2(\mu)$. □

Corollary 4.5. *$H^2(\mu) = L^2(\mu)$ if and only if* $\operatorname{dist}(1, H^2_0(\mu)) = 0$, *where $H^2_0(\mu)$ is the closure of* $\operatorname{span}\{z^n : n \geq 1\}$ *in $L^2(\mu)$.*

Proof. Let $H^2(\mu) = L^2(\mu)$, then $\bar{z} \in H^2(\mu)$, implies $\operatorname{dist}(1, H^2_0(\mu)) = \operatorname{dist}(\bar{z}, H^2(\mu)) = 0$. On the other hand, if $\operatorname{dist}(1, H^2_0(\mu)) = 0$, then $\bar{z} \in H^2(\mu)$, and hence $H^2(\mu) = L^2(\mu)$. □

The quantity

$$\overset{2}{\operatorname{dist}}(1, H^2_0(\mu)) = \inf_{p \in \mathbb{P}^0_+} \int_{\mathbb{T}} |1 - p|^2 \, d\mu$$

is called the **Szegö infimum**, where $\mathbb{P}^0_+ = \operatorname{span}\{z^n : n \geq 1\}$.

It depends only on the absolutely continuous part of μ. Indeed, let $d\mu = wdm + d\mu_s$ be the Lebesgue decomposition of μ. Arguing as in Corollary 4.2, we have $H^2_0(\mu) =$

$H_0^2(\mu_a) \oplus L^2(\mu_s)$. We also use the elementary fact that if M_1 and M_2 are orthogonal subspaces of a Hilbert space H, then $P_{M_1 \oplus M_2} = P_{M_1} + P_{M_2}$. Therefore,

$$\begin{aligned}
\overset{2}{\text{dist}}(1, H_0^2(\mu)) &= \|P_{H_0^2(\mu)} \perp 1\|_{L^2(\mu)}^2 \\
&= \|(P_{H_0^2(\mu_a)} \oplus P_{L^2(\mu_s)}) \perp (1_a + 1_s)\|_{L^2(\mu)} \\
&= \|P_{H_0^2(\mu_a)} \perp 1_a\|_{L^2(\mu_a)}^2 \text{ (since } 1_s \in L^2(\mu_s)) \\
&= \inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 w dm.
\end{aligned}$$

The evaluation of Szegö infimum is intimately related to the multiplicative structure of H^2.

4.4 The inner-outer factorization

Recall that a function $f \in H^2$ is called **inner** if $|f| = 1$ a.e. on $\mathbb{T}$. On the other hand, $f \in H^2$ is called **outer** if $E_f = H^2$.

Theorem 4.6. (V. Smirnov, 1928) *Let $f \in H^2$ and $f \not\equiv 0$, then there exists an inner function $f_{inn} \in H^2$ and an outer function $f_{out} \in H^2$ such that $f = f_{inn} f_{out}$. Moreover, this factorization is unique and $E_f = f_{inn} H^2$.*

Proof. Note that $E_f \subset H^2$, $E_f \neq \{0\}$, and E_f is not reducing, else $\bar{z} \in H^2$. Here, $E_f = \overline{\text{span}}\{z^n f : n \geq 0\} \subset H^2$. By Theorem 3.6, we have $E_f = \theta H^2$, where $|\theta| = 1$ a.e. m. Let $f_{inn} = \theta$, then $f = \theta g$, where $g \in H^2$. We claim $E_g = H^2$. Let $h \in H^2$. Since $E_f = \theta H^2$ and $\theta h \in \theta H^2$, there exists $p_n \in \mathbb{P}_+$ such that $p_n \theta g = p_n f \to \theta h$ in L^2. But, multiplication by an inner function is an isometry, we get

$$\|p_n g - h\|_2 = \|\theta(p_n g - h)\|_2 \to 0.$$

Hence, $E_g = H^2$. Here $g = f_{out}$ is desired outer function.

Uniqueness: Take $f = f_1 f_2$, where f_1 is inner and f_2 is outer. As f_1 is inner, $h \mapsto f_1 h$ is an isometry, and hence as $E_{f_2} = H^2$, we get

$$f_{inn} H^2 = E_f = \overline{\text{span}}\{z^n f_1 f_2 : n \geq 0\} = f_1 \overline{\text{span}}\{z^n f_2 : n \geq 0\} = f_1 H^2.$$

By the uniqueness of the representing inner function of the simply invariant space E_f

(cf. Theorem 3.6 and Corollary 3.7), we get $f_{inn} = \lambda f_1$ with $|\lambda| = 1$, and $\lambda f_1 f_{out} = f_1 f_2$ implies $f_{out} = \bar{\lambda} f_2$. □

4.5 Arithmetic of inner functions

Definition 4.7. Let θ_1, θ_2 be two inner functions in H^2. We say θ_1 divides θ_2 if $\frac{\theta_2}{\theta_1} \in H^2$.

Equivalently, θ_1 divides θ_2 if and only if $\theta_1 H^2 \supset \theta_2 H^2$. For this, if $\theta_2 = \theta\theta_1$, then θ is necessarily inner, and $\theta_2 H^2 = \theta_1 \theta H^2 \subset \theta_1 H^2$, since $\theta H^2 \subset H^2$. On the other hand, if $\theta_1 H^2 \supset \theta_2 H^2$, then we get $\theta_2 \in \theta_1 H^2$ implies $\theta = \frac{\theta_2}{\theta_1} \in H^2$.

We deduce the following two elementary properties:

Theorem 4.8. *Let $\theta = \gcd\{\theta_1, \theta_2\}$, the greatest common divisor of θ_1 and θ_2. Then*

(i) span $\{\theta_1 H^2, \theta_2 H^2\} = \theta H^2$

(ii) $\theta_1 H^2 \cap \theta_2 H^2 = \tilde{\theta} H^2$, where $\tilde{\theta} = \operatorname{lcm}\{\theta_1, \theta_2\}$.

Proof. (i) $\theta_k H^2 \subset \operatorname{span}\{\theta_1 H^2, \theta_2 H^2\} = \theta H^2\,;\ k = 1, 2$ for some inner function θ (by Beurling's theorem) implies θ divides $\theta_k\,;\ k = 1, 2$. Let θ' be another divisor of $\theta_k :\ k = 1, 2$. Then $\theta' H^2 \supset \theta_k H^2$, and hence $\theta' H^2 \supset \operatorname{span}\{\theta_k H^2;\ k = 1, 2\} = \theta H^2$. This implies θ' divides θ and thus $\theta = \gcd\{\theta_k;\ k = 1, 2\}$. The proof of (ii) is similar to (i). □

Definition 4.9. Let $\{\theta_i : i \in I\}$ be a family of inner functions.

(i) $\theta = \gcd\{\theta_i : i \in I\}$ if θ divides each θ_i, and θ is divisible by every other inner function that divides each θ_i.

(ii) $\theta = \operatorname{lcm}\{\theta_i : i \in I\}$ if each θ_i divides θ and θ divides every other inner function that is divisible by each θ_i

Convention: In case the gcd or the lcm does not exist, we write $\gcd\{\theta_i :\ i \in I\} = 1$ and $\operatorname{lcm}\{\theta_i : i \in I\} = 0$.

Corollary 4.10. span $\{\theta_i \in H^2 : i \in I\} = \theta H^2$, *where* $\theta = \gcd\{\theta_i : i \in I\}$ *and* $\cap\, \theta_i H^2 = \tilde{\theta} H^2$, *where* $\tilde{\theta} = \operatorname{lcm}\{\theta_i : i \in I\}$.

Corollary 4.11. *Let F be a proper subset of H^2. Then $\overline{\operatorname{span}}\{z^n F : n \geq 0\} = \theta H^2$, where $\theta = \gcd\{f_{inn} : f \in F \setminus \{0\}\}$, and f_{inn} stands for inner factor of f.*

Proof. We have $\overline{\operatorname{span}}\{z^n F : n \geq 0\} = \overline{\operatorname{span}}\{f_{inn} H^2 : f \in F \setminus \{0\}\}$. (By Smirnov's theorem). By applying Corollary 4.10 we get the required. □

4.6 Characterization of outer functions

Theorem 4.12 (Integral Maximum Principle)**.** *Let $f \in H^2$. Then the following are equivalent:*

(i) f is outer

(ii) f is a divisor of the space H^2, i.e. if $g \in H^2$ and $\frac{g}{f} \in L^2$, then $\frac{g}{f} \in H^2$.

Proof. (ii) $\Longrightarrow$ (i): Let $f = f_{inn} f_{out}$ be an inner-outer factorization of f. Then $\bar{f}_{inn} = \frac{1}{f_{inn}} = \frac{f_{out}}{f} \in L^2$ because of $f_{inn} \in H^2 \subset L^2$. By (ii), we get $\bar{f}_{inn} \in H^2$. But $f_{inn} \in H^2$ implies $\bar{f}_{inn} = \lambda$ (constant) with $|\lambda| = 1$. Hence $f = \bar{\lambda} f_{out}$.
(i) $\Longrightarrow$ (ii): Given f is outer, we have $E_f = H^2$. Since $1 \in H^2$, there exists $p_n \in \mathbb{P}_+$ such that $p_n f \to 1$ in L^2. Let $g \in H^2$ and $h = \frac{g}{f} \in L^2$. Then

$$\int_{\mathbb{T}} |p_n g - h| = \int_{\mathbb{T}} |p_n f - 1||h| \leq \|p_n f - 1\|_2 \|h\|_2 \to 0 \text{ as } n \to \infty. \tag{4.6.1}$$

But $p_n g \in H^2$, implies $\widehat{(p_n g)}(k) = 0$ if $k < 0$. Since $\varphi \mapsto \hat{\varphi}(k)$ is continuous linear functional on $L^1(\mathbb{T})$ for each k, by (4.6.1) we get $(\hat{h})(k) = 0$, $\forall\, k < 0$. Thus $h \in H^2$. □

Corollary 4.13. *If two outer functions f_1 and f_2 verify $|f_1| = |f_2|$ a.e. on $\mathbb{T}$, then $f_1 = \lambda f_2$ where $|\lambda| = 1$.*

Proof. Since f_2 is outer, $f_1 \in H^2$, and $|\frac{f_1}{f_2}| = 1 \in L^2$, by Theorem 4.12, we get $\frac{f_1}{f_2} \in H^2$. In the similar way $\overline{\frac{f_1}{f_2}} = \frac{f_2}{f_1} \in H^2$ implies $\frac{f_1}{f_2} = \lambda$ (constant) and hence $f_1 = \lambda f_2$ with $|\lambda| = 1$. Thus, an outer function is completely defined by its modulus. □

Corollary 4.14. *Let $w \geq 0$, $w \in L^1(\mathbb{T})$. If there exists $f \in H^2$ such that $|f|^2 = w$ a.e. $\mathbb{T}$, then there exists a unique outer function $f_0 \in H^2$ such that $|f_0|^2 = w$ a.e. $\mathbb{T}$.*

(Hint: By Smirnov theorem, $f = f_{inn} f_{out}$ etc.)

Corollary 4.15. *If $f \in H^2(\mathbb{T})$ is simultaneously inner and outer then f is constant.*

Proof. Since $f \in H^2(\mathbb{T})$ is inner $|f| = 1$ and hence $1/f = \overline{f} \in H^2(\mathbb{T})$ by the Theorem 4.12. Since $f, \overline{f} \in H^2(\mathbb{T})$ hence f is constant. □

4.7 Szegö infimum and the F. & M. Riesz theorem

Here we consider two theorems in two different settings by using the fact that in an orthogonal complement of the analytic polynomials $\mathbb{P}_+$ the absolute component of a measure is only important.

Theorem 4.16. (Szegö and Kolmogorov) *Let μ be a finite Borel measure on $\mathbb{T}$ with Lebesgue decomposition $d\mu = wdm + d\mu_s$, where $w \in L^1_+(\mathbb{T})$. Then*

(i) either there does not exist any $f \in H^2$ such that $|f|^2 = w$ a.e. m, then

$$\inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |1-p|^2 d\mu = 0.$$

(ii) or there exists (unique) $f \in H^2$ such that $|f|^2 = w$ a.e. m, and f is outer, then

$$\inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |1-p|^2 d\mu = |\hat{f}(0)|^2.$$

Proof. (ii)We know that the Szegö infimum I will satisfy

$$\begin{aligned} I^2 = \overset{2}{\text{dist}}(1, H^2_0(\mu)) &= \overset{2}{\text{dist}}(1, H^2_0(\mu_a)) \\ &= \inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |1-p|^2 wdm. \end{aligned}$$

Given that $|f|^2 = w$ a.e. m, and f is outer. Hence

$$I^2 = \inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |f - pf|^2 dm.$$

As f is an outer function, we can verify that $\overline{\text{span}}\{z^n f : n \geq 1\} = zH^2$. Hence $I = \text{dist}_{H^2}(f, zH^2)$. Note that $f = \sum\limits_{n\geq 0} \hat{f}(n)z^n = \hat{f}(0) + g$, where $g \in zH^2$. Since $\hat{f}(0) \perp zH^2$, it follows that $I = \text{dist}_{H^2}(\hat{f}(0), zH^2) = |\hat{f}(0)|$.
(i). Next, we consider the invariant space $E_a = H^2_0(\mu_a)$. If $zE_a \neq E_a$, then there exists θ such that $E_a = \theta H^2$ with $|\theta|^2 w \equiv 1$. But $z \in E_a$ and hence $z = \theta f$ for some $f \in H^2$. This implies that $|f|^2 = \frac{1}{|\theta|^2} = w$ (since $|z| = 1$), and this leads to case (ii). Hence, case (i) is possible only if $zE_a = E_a$. But, then $E_a = L^2(\mu_a)$ by Remark 4.3(i). Hence $\text{dist}(1, H^2_0(\mu)) = 0$, since $1 \in L^2(\mu_a) = H^2_0(\mu_a)$. □

Theorem 4.16 becomes completely explicit once we identify the outer function f with prescribed boundary modulus. In particular, if $\log w \in L^1(\mathbb{T})$ then the outer function

$$f(z) = \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,\log w(\zeta)\,dm(\zeta)\Big)$$

satisfies $|f|^2 = w$ a.e. on $\mathbb{T}$, and

$$|\widehat{f}(0)|^2 = |f(0)|^2 = \exp\Big(\int_{\mathbb{T}} \log w\,dm\Big).$$

Conversely, if $\log w \notin L^1(\mathbb{T})$ then no such outer $f \in H^2$ exists.

Corollary 4.17 (Szegő's formula for the distance)**.** *Let $\mu = wdm + \mu_s$ be a finite Borel measure on $\mathbb{T}$ with $w \in L^1_+(\mathbb{T})$. Then*

$$\inf_{p\in\mathbb{P}^0_+} \int_{\mathbb{T}} |1-p|^2\,d\mu = \begin{cases} \exp\Big(\int_{\mathbb{T}} \log w\,dm\Big), & \log w \in L^1(\mathbb{T}), \\ 0, & \log w \notin L^1(\mathbb{T}). \end{cases}$$

Remark 4.18 (Prediction-theoretic interpretation)*.* If $(X_n)_{n\in\mathbb{Z}}$ is a (centered) stationary process whose spectral measure is $d\mu = wdm + \mu_s$, then the quantity on the left-hand side of Theorem 4.17 is exactly the *optimal mean-square prediction error* for X_0 from the "past" $\{X_{-1}, X_{-2}, \dots\}$. Thus Szegő's formula identifies the prediction error with the geometric mean of the spectral density.

F. & M. Riesz result is an important consequence of Helson's theorem. For that, we need to recall an important result related to the Radon-Nikodym derivative.

Let $|\mu|$ be the total variation measure of a complex-valued Borel measure μ on $\mathbb{T}$, i.e.

$$|\mu|(\sigma) = \sup\Big\{\sum_{i\in I} |\mu(\sigma_i)| : \{\sigma_i\}_{i\in I} \text{ is a partition of } \sigma \text{ in } \mathcal{B}(\mathbb{T})\Big\}.$$

Suppose μ is absolutely continuous with respect to a positive measure λ on $\mathcal{B}(\mathbb{T})$. Then there exists $\varphi \in L^1(\lambda)$ (the Radon-Nikodym derivative of μ with respect to λ) such that

$$|\mu|(\sigma) = \int_{\sigma} |\varphi| d\lambda.$$

Theorem 4.19. (F. & M. Riesz, 1916) *Let μ be a complex-valued Borel measure on $\mathbb{T}$ such that*

$$\int_{\mathbb{T}} z^n d\mu = 0,\ \forall\, n \geq 1.$$

Then $\mu \ll m$ and $d\mu = h\,dm$, where $h \in H^1 = \{f \in L^1(\mathbb{T}) : \hat{f}(k) = 0,\ k < 0\}$.

Note that, a measure μ that satisfies $\int\limits_{\mathbb{T}} \bar{z}^n d\mu = 0$ for $n < 0$ will be called **analytic measure.**

Proof. It follows that $\mu \ll |\mu|$. Let $g \in L^1(|\mu|)$ be the corresponding Radon-Nikodym derivative of μ with respect to $|\mu|$. We claim that $|g| = 1$ a.e. μ. For $\delta > 0$, set $\sigma = \{t : |g(t)| < 1 - \delta\}$. Then $|\mu|(\sigma) = \int\limits_{\sigma} |g|d|\mu| \le (1-\delta)|\mu|(\sigma)$. This implies $|\mu|(\sigma) = 0$. Similarly, the case $\sigma' = \{t : |g(t)| > 1 - \delta\}$. This proves the claim. As a consequence of the Corollary 4.2, we get

$$H_0^2(|\mu|) = H^2(|\mu|_a) \oplus L^2(|\mu|_s). \tag{4.7.1}$$

But $|g| = 1$ a.e. $|\mu|$ implies $\bar{g} \in L^2(|\mu|)$, and

$$\langle z^n, \bar{g}\rangle_{L^2(|\mu|)} = \int_{\mathbb{T}} z^n g d|\mu| = \int_{\mathbb{T}} z^n d\mu = 0,\ n \ge 1.$$

In other words, $\bar{g} \perp z^n$, $n \ge 1$ in the Hilbert space $L^2(|\mu|)$, and hence $\bar{g} \perp H_0^2(|\mu|)$. In view of (4.7.1), we obtain $\bar{g} \perp H_0^2(|\mu|_s)$. Next, by construction, $|g| = 1$ a.e. $|\mu|$, which implies $|g| = 1$ a.e. $|\mu|_s$. This is impossible (since $\bar{g} \perp H_0^2(|\mu|_s)$), unless $|\mu|_s = 0$. Finally, $\mu \ll |\mu|$ implies

$$\mu(\sigma) = \int_{\sigma} g d|\mu| = \int_{\sigma} g d|\mu|_a = \int_{\sigma} g w dm$$

for each $\sigma \in \mathcal{B}(\mathbb{T})$. That is $\mu \ll m$ with Radon-Nikodym derivative $h = gw \in L^1(\mathbb{T})$, and

$$\hat{h}(k) = \int_{\mathbb{T}} \bar{z}^k h dm = \int_{\mathbb{T}} \bar{z}^k g w dm = \int_{\mathbb{T}} \bar{z}^k d\mu = 0 \quad \text{if } k \le -1.$$

Hence $h \in H^1$. □

Question 4.20. *

For $g \in L^1(\mathbb{T})$, define $g_f = \overline{\text{span}}\{z^n g : n \ge 0\}|_{L^1(\mathbb{T})}$. Characterize all possible $g \in L^1(\mathbb{T})$ such that $\inf\limits_{p \in P_+^0} \|1 - p\, g\|_1 = 0$.

4.8 Guided examples and exercises

The first block collects short worked examples that may be used as templates for later proofs about inner, outer, and singular factors.

Worked examples

Example 4.21. $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda} z}$ where $\lambda \in \mathbb{D}$ is an inner.

Proof. $b_\lambda = \lambda - z\sum_{n\geq 0}\overline{\lambda}^n z^n (|z|=1)$ and clearly $\widehat{b}_\lambda(k) = 0$ for $k<0$, and $\sum_{k\geq 0}|\widehat{b}_\lambda(k)|^2 < \infty$; hence $b_\lambda \in H^2(\mathbb{T})$. Moreover, for $|z|=1$ we have $|\lambda - z| = |\overline{\lambda} - \overline{z}| = |1-\overline{\lambda}z|$, thus $|b_\lambda(z)| = 1$. ☐

Example 4.22. $f = \Pi_{k=1}^N b_{\lambda_k}$ is an inner.

Proof. For $f, g \in H^\infty$ we have $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ hence $H^\infty.H^\infty \subset H^\infty$, a product of inner function is inner. ☐

Example 4.23. $S_{\zeta,\alpha} = \exp(\frac{-a(\zeta+z)}{\zeta - z})$ where $a>0, \zeta\in\mathbb{T}$.

Proof. As Re $\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{1-|z|^2}{|\zeta-z|^2} \geq 0$ for any $\zeta \in \mathbb{T}$ and $|z|<1$, we obtain that $|S_{\zeta,a}| = 1$ on $\mathbb{T}$. Moreover for every $n>0$ we have $\widehat{S}_{\zeta,a}(-n) = \int_{\mathbb{T}} z^n S_{\zeta,a}(z)dm = \lim_{r\to 1}\int_{\mathbb{T}} f_r(z)dm = 0$ where $f(z) = z^n S_{\zeta,a}(z)$ and $f_r(z) = f(rz), 0\leq r<1$ ($\widehat{f}_r(0) = 0$ since f_r is analytic in $|z|<1/r$ and $f_r(0)=0$.) ☐

Example 4.24. $f = \prod_{k=1}^N S_{\zeta_k, a_k}$ where $a_k > 0$ $\zeta_k \in \mathbb{T}$.

Proof. See the proof of (ii). ☐

Examples related to the outer functions you will get in Chapter 6, Subsection 6.2.

Core exercises

Exercise 4.25. For every $f \in L^2$ prove that $f\cdot H^\infty(\mathbb{T}) \subset E_f = \overline{\text{span}}\{f, zf, z^2 f, \dots\}$.

Proof. Clearly $f\mathcal{P}_a \subset E_f$, where $\mathcal{P}_a$ denotes the analytic polynomials. Thus it is enough to prove that $(f\mathcal{P}_a)^\perp \subset (fH^\infty)^\perp$ (orthogonal complements taken in L^2). Let $g \in (f\mathcal{P}_a)^\perp$; equivalently, $\int_{\mathbb{T}} \overline{g} f p\, dm = 0$ for every polynomial $p \in \mathcal{P}_a$. Now let $h \in H^\infty$. Since $\overline{g}f \in L^1(\mathbb{T})$ and Fejér polynomials of h converge to h in the weak-$*$ topology $\sigma(L^\infty, L^1)$, we obtain $\int_{\mathbb{T}} \overline{g} f h\, dm = 0$. Hence $g \in (fH^\infty)^\perp$, as required. ☐

Example 4.26. If $f \in H^2(\mathbb{T})$ such that $1/f \in H^\infty(\mathbb{T})$, then f is an outer.

Proof. By the exercise 4.25, $1 = f\cdot 1/f \in E_f$ hence $E_f = H^2(\mathbb{T})$. ☐

Exercise 4.27. Let $f, g \in L^2(\mathbb{T})$ (thus $fg \in L^1(\mathbb{T})$). Show that for every $n \in \mathbb{Z}, \overline{fg}(n) = \sum_{k\in\mathbb{Z}} \overline{g}(k)\overline{f}(n-k)$; the series converges absolutely.

Proof. By Cauchy Schwarz's inequality $\|f(g-g')\| \leq \|f\|_2 \|g-g'\|_2$, the multiplication $M_g f = fg$ is continuous $L^2(\mathbb{T}) \to L^1(\mathbb{T})$. Moreover the Fourier series $g = \sum_{k\in\mathbb{Z}} \widehat{g}(k) z^k$ converges for the norm of $L^2(\mathbb{T})$. Hence $fg = \sum_{k\in\mathbb{Z}} \widehat{g}(k) z^k f$ converges in $L^1(\mathbb{T})$, which implies $\widehat{fg}(n) = \sum_{k\in\mathbb{Z}} \widehat{g}(k)\widehat{(z^k f)}(n)$. The calculation follows from $\widehat{(z^k f)}(n) = \widehat{f}(n-k)$. ☐

Exercise 4.28. Let $f = f_{in} f_{out} \in H^2(\mathbb{T})$. Show that $\sup\{|\widehat{g}(0)| : g \in H^2(\mathbb{T}), |g| \leq |f| \text{ a.e. on } \mathbb{T}\} = |\widehat{f}_{out}(0)|$

Proof. From the previous exercise $\widehat{\varphi\psi}(0) = \widehat{\varphi}(0)\widehat{\psi}(0)$ for all $\varphi, \psi \in H^2(\mathbb{T})$. Moreover for every inner function h, we have $|\widehat{h}(0)| \leq \|h\|_1 = 1$. Given $g \in H^2(\mathbb{T},)|g| \leq |f|$, which implies $|\widehat{g}(0) = |\widehat{g}_{in}(0)\widehat{g}_{out}(0)| \leq |\widehat{g}_{out}(0)|$. Then by Theorem 4.16

$$|\widehat{g}(0)|^2 \leq |\widehat{g}_{out}(0)|^2 = \inf_{p \in \mathcal{P}_a} \int_{\mathbb{T}} |1-p|^2 |g|^2 dm \leq \inf_{p \in \mathcal{P}_a} \int_{\mathbb{T}} |1-p|^2 |f|^2 dm = |\widehat{f}_{out}(0)|^2$$

□

Chapter 5

Canonical factorization in $H^p(\mathbb{D})$

This chapter establishes the canonical factorization theory in Hardy spaces on the disk. We revisit Fourier series and identify $H^p(\mathbb{D})$ with its boundary function space on $\mathbb{T}$, then prove Jensen's formula/inequality and boundary uniqueness results. We construct Blaschke products, develop non-tangential boundary limits (Fatou theory), and culminate with the Riesz–Smirnov factorization into Blaschke, singular inner, and outer factors, together with approximation consequences.

Learning objectives.

- Identify $H^p(\mathbb{D})$ with its boundary-value realization and track what changes with the exponent p.
- Prove Jensen-type formulas and use them to control zero sets and boundary growth.
- Assemble the canonical factorization and understand how each factor records different analytic data.

Key ideas.

- The passage from interior growth to boundary data is the foundational move in Hardy-space theory.
- Jensen's formula turns zeros into logarithmic averages and is therefore the quantitative core of canonical factorization.
- Blaschke, singular inner, and outer factors are not merely convenient pieces; they are the three irreducible sources of Hardy-space behaviour.

Example 5.1 (A convergent Blaschke pattern)**.** The sequence $a_n = 1 - 2^{-n}$ satisfies $\sum_n (1 - |a_n|) < \infty$. Hence the infinite product

$$B(z) = \prod_{n \geq 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

converges to an inner function whose zero set is exactly $\{a_n\}$. This is the prototypical mechanism by which zero distributions enter canonical factorization.

In this chapter we develop the canonical factorization of functions in H^p on the open unit disk into three components: a Blaschke product, a singular inner function, and an outer function given through its Schwarz–Herglotz representation. This factorization clarifies many of the structural questions raised earlier, including approximation problems and the Szegő infimum.

Definition 5.2. Let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and denote by $\mathrm{Hol}(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$. For $p > 0$, set

$$H^p(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^p}^p = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^p \, dt < \infty \right\},$$

and

$$H^\infty(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

Here dt denotes normalized Lebesgue measure on $\mathbb{T}$.

For $p \geq 1$, set $L^p = L^p([0, 2\pi], dt)$ and

$$H^p = \{f \in L^p : \widehat{f}(k) = 0 \text{ for } k < 0\}.$$

The spaces $H^p(\mathbb{D})$ and H^p will later be canonically identified through boundary values; when no ambiguity can arise, we refer to both simply as Hardy spaces.

5.0.1 Properties of H^p spaces

(i) $H^p(\mathbb{D})$ is a linear space.

(ii) $f \longmapsto \|f\|_{H^p}$ is a norm if $p \geq 1$.

(iii) $H^p(\mathbb{D}) \subset H^q(\mathbb{D})$ if $p > q$.

(iv) For $p = 2$, let $f \in \mathrm{Hol}(\mathbb{D})$, and

$$f(z) = \sum_{n\geq 0} \hat{f}(n)z^n,\ \hat{f}(n) \in \mathbb{C}.$$

By Parseval's identity

$$\int_0^{2\pi} |f(re^{it})|^2 dt = \sum_{n\geq 0} |\hat{f}(n)|^2 r^{2n},\ 0 \leq r < 1$$

and we have

$$\sup_{0\leq r<1} \int_0^{2\pi} |f(re^{it})|^2 dt = \sum_{n\geq 0} |\hat{f}(n)|^2.$$

Thus for $f \in \mathrm{Hol}(\mathbb{D})$, we have $f \in H^2(\mathbb{D})$ if and only if $\sum_{n\geq 0} |\hat{f}(n)|^2 < \infty$.

(v) If $1 \leq p \leq \infty$, H^p is a Banach space, and $0 < p < 1, H^p$ is a complete metric space [13](p. 37). If $p < 1$, then $\|.\|_p$ is not a true norm, in fact H^p is not normable. However the expression $d(f, g) = \|f - g\|_p^p$ defines a metric on H^p if $p < 1$.

Example 5.3. The function f(z)=$\frac{1}{1-z}$ is analytic on $\mathbb{D}$ but is not in $H^2(\mathbb{D})$.

Proof. Since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, the coefficients of f are not square-summable. □

For $f \in H^\infty, \|f\|^2 = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) \leq \|f\|_\infty^2 < \infty \implies f \in H^2$, hence $H^\infty \subset H^2$.

Example 5.4. The inclusion $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$ is strict since the function f(z)=log $\frac{1}{1-z}$ is an unbounded analytic function on $\mathbb{D}$ but it belongs to $H^2(\mathbb{D})$, because it has a Taylor series:

$$\log \frac{1}{1-z} = \sum_{n\geq 1} \frac{z^n}{n}$$

has square summable coefficients.

5.1 A Revisit to Fourier Series

The functions in $L^p[0, 2\pi]$ can be thought of as functions on $(0, 2\pi)$, which can be extended periodically to the real line $\mathbb{R}$.

Lemma 5.5. *Let $f \in L^1[0, 2\pi]$, $g \in L^p[0, 2\pi]$, $1 \leq p \leq \infty$. Then*

(i) for almost every $x \in (0, 2\pi)$, $y \longmapsto f(x - y)g(y)$ is integrable on $(0, 2\pi)$.

(ii) $f * g(x) = \int_0^{2\pi} f(x-y)g(y)dy$ *is well defined and belongs to* $L^p[0, 2\pi]$.

(iii) $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof. Note that $(x, y) \longmapsto f(x-y)g(y)$ is measurable, and by Fubini's theorem $|f*g(x)| \leq \int |f(x-y)||g(y)|dy < \infty$ a.e. x. By Minkowski integral inequality,

$$\Big\| \int f(x-y)g(y)dy \Big\|_p \leq \int \|f(x-y)g(y)\|_p dy = \|g\|_p \|f\|_1.$$

Further, if $f \in L^1(0, 2\pi)$ and $\hat{f}(n) = \int_0^{2\pi} f(t)e^{-int}dt$, then $\widehat{(f * g)}(n) = \hat{f}(n)\hat{g}(n)$, whenever $g \in L^p$ and $1 \leq p \leq \infty$ (using Fubini's theorem). □

5.1.1 Approximation identity (or good kernel)

(i) If a family $(E_\alpha) \subset L^1$ satisfies

(a) $\sup_\alpha \|E_\alpha\|_1 < \infty$

(b) $\lim_\alpha \hat{E}_\alpha(n) = 1$,

then $\lim_\alpha \|f - f * E_\alpha\|_p = 0$ for $f \in L^p (1 \leq p < \infty)$. This is still true for $p = \infty$, if $f \in C(\mathbb{T})$ (called **approximate identity of** L^p.)

(ii) If $(E_\alpha) \subset L^1$ satisfies

(a) $\sup_\alpha \|E_\alpha\|_1 < \infty$

(b) $\lim_\alpha \int_0^{2\pi} E_\alpha dx = 1$

(c) $\lim_\alpha \sup_{\delta < |x| < \pi} |E_\alpha(x)| = 0$, $\forall\, \delta > 0$.

then conditions of (a) and (b) of (i) is satisfied and we get $\lim_\alpha \|f - f * E_\alpha\|_p = 0$.

5.1.2 Dirichlet, Fejér and Poisson kernels

(i) Dirichlet kernel

$$D_m = \sum_{k=-m}^{m} e^{ikt} = \frac{\sin(m + \frac{1}{2})t}{\sin(t/2)}.$$

(ii) Fejer kernel

$$\Phi_n(t) = \frac{1}{n+1} \sum_{m=0}^{n} D_m = \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n+1}\Big) e^{ikt} = \frac{1}{n+1} \Big(\frac{\sin \frac{n+1}{2} t}{\sin(t/2)}\Big)^2.$$

(iii) Poisson kernel

$$P_r(t) = P(re^{it}) = \frac{1-r^2}{|1-re^{it}|^2} = \sum_{k\in\mathbb{Z}} r^{|k|}e^{ikt},\ 0 \le r < 1.$$

Result: If $f \in L^1$, then

1. $f * D_m(t) = \sum_{k=-m}^{m} \hat{f}(k)e^{ikt} = S_m(f;t)$ (Partial Fourier series sums of f)
2. $f * \Phi_n(t) = \sum \hat{f}(j)\Big(1 - \dfrac{|j|}{n+1}\Big)e^{ijt} = \dfrac{1}{n+1}\sum_{m=0}^{n} S_m(f;t)$ (Arithmetic mean of partial sum of Fourier series of f)
3. $f * P_r(t) = \sum_{k\in\mathbb{Z}} \hat{f}(k)r^{|k|}e^{ikt}$, $0 \le r < 1$.
4. $(\Phi_n)_{n\ge1}$ and $(P_r)_{0\le r<1}$ are good kernels, and $\|P_r\|_1 = \|\Phi_n\|_1 = 1$.
5. $P_r * P_{r'} = P_{rr'}$ for $0 \le r, r' < 1$ (semigroup property).

Corollary 5.6. *If $f \in L^p$, $1 \le p < \infty$, then $\lim_{n\to\infty} \|f - f * \Phi_n\|_p = 0$. Hence trigonometric polynomials are dense in L^p. (Hint: This follows from the property of the good kernel.)*

The same is true for $p = \infty$, if $f \in C(\mathbb{T})$.

Corollary 5.7. *If $f \in L^1$, $\hat{f}(n) = 0$, $\forall\, n \in \mathbb{Z}$, then $f = 0$.*

Notations: For $f \in L^1$, set $f_r = f * P_r$, $0 \le r < 1$.
For $f \in \mathrm{Hol}(\mathbb{D})$, we set $f_{(r)}(z) = f(rz)$, if $|z| < \frac{1}{r}$, $0 \le r < 1$. Clearly $f_{(r)}$ is analytic in bigger domain: $|z| < \frac{1}{r} < 1 + \epsilon$.

Corollary 5.8. *If $0 \le r < \rho < 1$ and $f \in L^p$, $1 \le p < \infty$, then $\lim_{r\to1} \|f_r - f\|_p = 0$. Moreover, $\|f_r\|_p \le \|f_\rho\|_p \le \|f\|_p$(using maximum modulus principle).*

If $f \in \mathrm{Hol}(\mathbb{D})$, then $\|f_{(r)}\|_p \le \|f_{(\rho)}\|_p$ and the limit (possible infinite) $\lim_{r\to1} \|f_{(r)}\|_p \le \infty$, exists. In fact, $\lim_{r\to1} \|f_{(r)}\|_p = \|f\|_{H^p(\mathbb{D})}$ if $f \in H^p(\mathbb{D})$. (It follows due to P_r is a good kernel.)

5.2 Identification of $H^p(\mathbb{D})$ with $H^p(\mathbb{T})$

Section roadmap.

- Start from radial dilations and Poisson integrals to produce boundary values.

- Use Fourier support to identify which L^p boundary functions come from analytic functions.
- Record carefully where the argument changes between $p \geq 1$, $0 < p < 1$, and $p = \infty$.

Theorem 5.9 (Identification of $H^p(\mathbb{D})$ with boundary functions)**.** *Let $1 \leq p \leq \infty$, and let $f \in H^p(\mathbb{D})$. For $0 < r < 1$ set*

$$f_{(r)}(e^{it}) := f(re^{it}), \qquad t \in [0, 2\pi).$$

Then there exists a unique *function $\tilde{f} \in L^p(\mathbb{T})$ such that:*

(i) $f_{(r)} \to \tilde{f}$ in $L^p(\mathbb{T})$ as $r \uparrow 1$ whenever $1 \leq p < \infty$; if $p = \infty$, the convergence holds in the weak-$*$ topology of $L^\infty(\mathbb{T})$;

(ii) $\tilde{f} \in H^p(\mathbb{T})$, i.e. $\widehat{\tilde{f}}(n) = 0$ for all $n < 0$;

(iii) for every $0 < r < 1$ we have

$$f_{(r)} = \tilde{f} * P_r \quad \text{on } \mathbb{T},$$

equivalently, f is the Poisson extension of its boundary function $\tilde{f}$;

(iv) the mapping $f \mapsto \tilde{f}$ is an isometry:

$$\|f\|_{H^p(\mathbb{D})} = \|\tilde{f}\|_{L^p(\mathbb{T})}.$$

The function $\tilde{f}$ is called the ***(radial) boundary function*** *(or boundary limit) of f.*

Proof. Write $f(z) = \sum_{n \geq 0} a_n z^n$. For each $0 < r < 1$ we have

$$f_{(r)}(e^{it}) = \sum_{n \geq 0} a_n r^n e^{int},$$

so $(f_{(r)})_{0<r<1}$ is a bounded family in $L^p(\mathbb{T})$, with

$$\sup_{0<r<1} \|f_{(r)}\|_{L^p} = \|f\|_{H^p(\mathbb{D})} < \infty.$$

Step 1: existence of a boundary limit. If $1 < p < \infty$, then $L^p(\mathbb{T})$ is reflexive; hence, by Banach–Alaoglu, there exists a sequence $r_k \uparrow 1$ such that $f_{(r_k)} \rightharpoonup \tilde{f}$ weakly in $L^p(\mathbb{T})$

for some $\tilde{f} \in L^p(\mathbb{T})$. If $p = 1$, we view $L^1(\mathbb{T})$ as a subspace of $\mathcal{M}(\mathbb{T}) = C(\mathbb{T})^*$; again by Banach–Alaoglu, along a subsequence $r_k \uparrow 1$ we obtain a weak-$*$ limit $\nu \in \mathcal{M}(\mathbb{T})$ of $f_{(r_k)}\, dm$. If $p = \infty$, we apply weak-$*$ compactness of the unit ball of $L^\infty(\mathbb{T})$ to obtain a weak-$*$ limit in L^∞.

Step 2: identification of Fourier coefficients. For each $n \in \mathbb{Z}$, the functional $\varphi \mapsto \widehat{\varphi}(n)$ is continuous on $L^p(\mathbb{T})$ when $1 \le p < \infty$, and on $\mathcal{M}(\mathbb{T})$ in the weak-$*$ topology. Therefore,

$$\widehat{\tilde{f}}(n) = \lim_{k\to\infty} \widehat{f_{(r_k)}}(n) = \lim_{k\to\infty} a_n r_k^n = \begin{cases} a_n, & n \ge 0, \\ 0, & n < 0. \end{cases}$$

(Here we interpret $a_n = 0$ for $n < 0$.) Hence $\tilde{f}$ has no negative Fourier modes and therefore $\tilde{f} \in H^p(\mathbb{T})$.

Step 3: Poisson integral identity and strong convergence. For $0 < r < 1$, the Poisson kernel satisfies $\widehat{P_r}(n) = r^{|n|}$, so convolution with P_r multiplies Fourier coefficients by $r^{|n|}$. Consequently,

$$\widehat{(\tilde{f} * P_r)}(n) = \widehat{\tilde{f}}(n)\, r^{|n|} = \begin{cases} a_n r^n, & n \ge 0, \\ 0, & n < 0, \end{cases}$$

which coincides with $\widehat{f_{(r)}}(n)$. Thus $f_{(r)} = \tilde{f} * P_r$ on $\mathbb{T}$ for every $0 < r < 1$.

If $1 \le p < \infty$, then $(P_r)_{0<r<1}$ is an approximate identity in $L^p(\mathbb{T})$, hence

$$\|f_{(r)} - \tilde{f}\|_{L^p} = \|\tilde{f} * P_r - \tilde{f}\|_{L^p} \longrightarrow 0 \qquad (r \uparrow 1).$$

This shows that $\tilde{f}$ does not depend on the chosen subsequence and yields the full L^p convergence. The case $p = \infty$ is analogous, with convergence in the weak-$*$ topology.

Step 4: isometry. For $1 \le p < \infty$, the L^p convergence gives

$$\|\tilde{f}\|_{L^p} = \lim_{r\uparrow 1} \|f_{(r)}\|_{L^p} = \|f\|_{H^p(\mathbb{D})}.$$

If $p = \infty$, weak-$*$ convergence yields $\|\tilde{f}\|_\infty \le \|f\|_{H^\infty}$, while the identity $f_{(r)} = \tilde{f} * P_r$ implies $\|f_{(r)}\|_\infty \le \|\tilde{f}\|_\infty$ and hence $\|f\|_{H^\infty} \le \|\tilde{f}\|_\infty$ after taking $\sup_r$. □

Convention. For $1 \le p \le \infty$ we will routinely identify a function $f \in H^p(\mathbb{D})$ with its boundary function $\tilde{f} \in H^p(\mathbb{T})$ given by Theorem 5.9. In particular, for every $0 < r < 1$,

$$f(re^{it}) = (\tilde{f} * P_r)(e^{it}) \qquad \text{for a.e. } t \in [0, 2\pi),$$

and the Fourier coefficients of $\tilde{f}$ coincide with the Taylor coefficients of f:

$$\widehat{\tilde{f}}(n) = a_n \quad (n \geq 0), \qquad \widehat{\tilde{f}}(n) = 0 \quad (n < 0),$$

where $f(z) = \sum_{n \geq 0} a_n z^n$.

Corollary 5.10. *For every $\xi \in \mathbb{D}$, the pointwise evaluation functional*

$$\varphi_\xi : H^1(\mathbb{D}) \to \mathbb{C}, \qquad \varphi_\xi(f) := f(\xi),$$

is continuous. Consequently, evaluation is continuous on $H^p(\mathbb{D})$ for every $1 \leq p < \infty$.

Proof. Let $f \in H^1(\mathbb{D})$ and let $\tilde{f} \in H^1(\mathbb{T})$ be its boundary function from Theorem 5.9. Writing $\xi = re^{it}$ with $0 \leq r < 1$, we have the Poisson representation

$$f(\xi) = f(re^{it}) = (\tilde{f} * P_r)(e^{it}) = \int_{\mathbb{T}} \tilde{f}(\zeta)\, P_\xi(\zeta)\, dm(\zeta),$$

where $P_\xi(\zeta) = \frac{1-|\xi|^2}{|\zeta-\xi|^2}$ is the Poisson kernel centered at ξ. Hence, by Hölder's inequality,

$$|f(\xi)| \leq \|\tilde{f}\|_{L^1(\mathbb{T})}\, \|P_\xi\|_{L^\infty(\mathbb{T})} = \|\tilde{f}\|_1\, \frac{1+|\xi|}{1-|\xi|}.$$

Since $\|\tilde{f}\|_1 = \|f\|_{H^1(\mathbb{D})}$, the estimate gives continuity of φ_ξ on H^1. □

Remark 5.11. Let $1 \leq p < \infty$. If $\tilde{f}_n \to \tilde{f}$ in $L^p(\mathbb{T})$, where $\tilde{f}_n, \tilde{f} \in H^p(\mathbb{T})$, and if $f_n, f \in H^p(\mathbb{D})$ denote their Poisson extensions, then $f_n \to f$ uniformly on every compact subset of $\mathbb{D}$.

Proof. Fix $0 < r < 1$. By Theorem 5.10, for every $\lambda \in \mathbb{D}$ with $|\lambda| \leq r$ we have

$$|f_n(\lambda) - f(\lambda)| \leq \|\tilde{f}_n - \tilde{f}\|_{L^p(\mathbb{T})}\, \|P_\lambda\|_{L^{p'}(\mathbb{T})} \leq C_r\, \|\tilde{f}_n - \tilde{f}\|_{L^p(\mathbb{T})},$$

where $p' = \frac{p}{p-1}$ (with the usual convention $p' = \infty$ for $p = 1$) and $C_r := \sup_{|\lambda| \leq r} \|P_\lambda\|_{L^{p'}(\mathbb{T})} < \infty$. Since $\|\tilde{f}_n - \tilde{f}\|_{L^p} \to 0$, the right-hand side tends to 0 uniformly in $|\lambda| \leq r$. Hence $f_n \to f$ uniformly on the closed disk $\{|\lambda| \leq r\}$, and therefore on every compact subset of $\mathbb{D}$. □

5.3 Jensen's formula and Jensen's inequality

See Figure 5.1 for a schematic illustration.

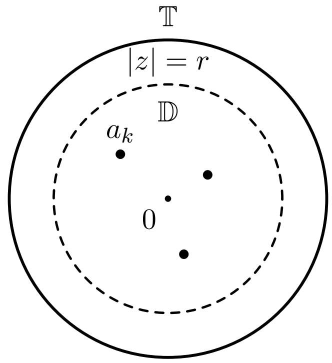

Figure 5.1: Geometry in Jensen's formula: zeros a_k inside a circle $|z| = r < 1$ and boundary averages on $|z| = r$.

Theorem 5.12 (Jensen's formula and Jensen's inequality)**.** *Let $f \in H^1(\mathbb{D})$ be nonzero, and let (λ_n) be the zeros of f in $\mathbb{D}$, listed with multiplicity. Then for every $0 < r < 1$,*

$$\log|f(0)| + \sum_{|\lambda_n|<r} \log\frac{r}{|\lambda_n|} = \int_0^{2\pi} \log|f(re^{it})|\,\frac{dt}{2\pi}. \tag{5.3.1}$$

In particular, letting $r \uparrow 1$ and writing $\tilde{f}$ for the boundary function of f, we obtain Jensen's inequality

$$\log|f(0)| + \sum_{n\geq 1} \log\frac{1}{|\lambda_n|} \leq \int_{\mathbb{T}} \log|\tilde{f}(\zeta)|\,dm(\zeta), \tag{5.3.2}$$

where the left-hand side may equal $-\infty$.

Proof. Fix $0 < r < 1$ and set $F_r(z) := f(rz)$, so that $F_r \in H^1(\mathbb{D})$ and

$$F_r(e^{it}) = f(re^{it}) \quad \text{for } t \in [0, 2\pi).$$

The zeros of F_r in $\mathbb{D}$ are precisely λ_n/r with $|\lambda_n| < r$ (counted with multiplicity); in particular, there are only finitely many such zeros.

Let

$$B_r(z) := \prod_{|\lambda_n|<r} b_{\lambda_n/r}(z), \qquad b_a(z) := \frac{|a|}{a}\,\frac{a - z}{1 - \overline{a}\,z} \quad (a \neq 0),$$

and interpret $b_0(z) = z$. Then B_r is a finite Blaschke product, so $|B_r(e^{it})| = 1$ for all t. Define

$$G_r(z) := \frac{F_r(z)}{B_r(z)}.$$

By construction, G_r is holomorphic and has no zeros in $\mathbb{D}$, hence $\log|G_r|$ is harmonic on

$\mathbb{D}$. Therefore, by the mean-value property,

$$\log|G_r(0)| = \int_0^{2\pi} \log|G_r(e^{it})| \, \frac{dt}{2\pi}. \tag{5.3.3}$$

Since $|B_r(e^{it})| = 1$, we have $\log|G_r(e^{it})| = \log|F_r(e^{it})| = \log|f(re^{it})|$. Moreover,

$$G_r(0) = \frac{F_r(0)}{B_r(0)} = \frac{f(0)}{\prod_{|\lambda_n|<r} \left|\lambda_n/r\right|} = f(0) \prod_{|\lambda_n|<r} \frac{r}{|\lambda_n|}.$$

Taking logarithms and substituting into (5.3.3) yields (5.3.1).

Finally, (5.3.2) follows by letting $r \uparrow 1$ in (5.3.1). Indeed, for each fixed n we have $\mathbf{1}_{\{|\lambda_n|<r\}} \uparrow 1$ as $r \uparrow 1$, so the sum on the left increases to $\sum_{n\geq 1} \log \frac{1}{|\lambda_n|}$ (possibly $+\infty$), while the right-hand side satisfies

$$\int_0^{2\pi} \log|f(re^{it})| \, \frac{dt}{2\pi} \longrightarrow \int_{\mathbb{T}} \log|\tilde{f}(\zeta)| \, dm(\zeta)$$

by Theorem 5.9 and standard approximation for log applied to the L^1 convergence of $f_{(r)}$ to $\tilde{f}$. This yields (5.3.2). □

Corollary 5.13 (Generalized Jensen's inequality)**.** *Let $g \in H^1$, $g \not\equiv 0$, and $|\xi| < 1$. Then*

$$\log|g(\xi)| \leq \int \frac{1-|\xi|^2}{|\xi - t|^2} \log|g(t)| dm(t). \tag{5.3.4}$$

Indeed, to begin with, we may assume that $g \in \mathrm{Hol}(\mathbb{D}_{1+\epsilon})$. Apply the previous result to the function

$$f(z) = g\Big(\frac{\xi - z}{1 - \bar{\xi} z}\Big),$$

and remark that Jacobian of this change of variable is $\frac{1-|\xi|^2}{|\xi - z|}$. (Hint: Put $s = \frac{\xi - t}{1 - \bar{\xi} t}$ etc.)

Remark 5.14. (Confrontation of two Jensen inequalities) Curiously, Jensen's inequality of Lemma 5.12 and Corollary 5.13 for the holomorphic functions is, in a way, the opposite of the fundamental inequality of convexity in real analysis, which also bears the name of Johan Jensen. In fact, the Jensen convexity inequality states that:

$$\varphi \int_{\mathbb{T}} g dm \leq \int_{\mathbb{T}} \varphi g dm$$

for any real integrable function g and any convex function $\varphi (\varphi'' > 0)$. Setting $g = \log|f|$

and $\varphi(x) = e^x$ we obtain the following:

$$\int_{\mathbb{T}} \log|f| dm \le \log \int_{\mathbb{T}} |f| dm = \log |\widehat{f}|(0)$$

5.4 The boundary uniqueness theorem

Corollary 5.15. *If $g \in H^1, g \not\equiv 0$, then $\log|g| \in L^1(\mathbb{T})$. In particular, if $g \in H^1$ and $m\{t \in \mathbb{T} : g(t) = 0\} > 0$, then $g \equiv 0$.*

Proof. Indeed, when $g \in H^1$ is realized on the disk $\mathbb{D}$, it has a Taylor expansion

$$g(z) = \sum_{k \ge n} \widehat{g}(k) z^k,$$

where $\widehat{g}(n) \ne 0$ and $n \ge 0$ is the multiplicity of the zero of g at 0. Applying Jensen's inequality to the function $f = g/z^n$, we obtain

$$\int_{\mathbb{T}} \log|g|\, dm = \int_{\mathbb{T}} \log|f|\, dm > -\infty.$$

Since $\log x \le x$ for $x > 0$, we also have

$$\int_{\mathbb{T}} \log|g|\, dm \le \int_{\mathbb{T}} |g|\, dm < \infty.$$

Hence $\log|g| \in L^1(\mathbb{T})$. It follows that if $m\{t \in \mathbb{T} : g(t) = 0\} > 0$, then

$$\int_{\mathbb{T}} \log|g|\, dm = -\infty,$$

which is possible only when $g \equiv 0$. □

Remark 5.16. The corollary is true for all $p > 0$. Proof for this using the MVT for harmonic function is done in the proof of Theorem 5.33.

Remark 5.17. Recall that we have seen the second statement of the above corollary for $f \in H^2$ using a completely different approach.

5.5 Blaschke products

See Figure 5.2 for a schematic illustration.

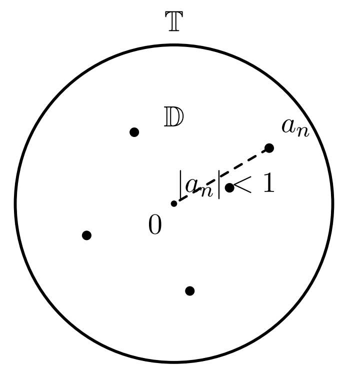

Figure 5.2: Zeros $(a_n) \subset \mathbb{D}$ of a Blaschke product. The Blaschke condition $\sum_n (1-|a_n|) < \infty$ controls how the zeros may accumulate at $\mathbb{T}$.

Lemma 5.18. (*Blaschke condition, interior uniqueness theorem*) *Suppose* $f \in \mathrm{Hol}(\mathbb{D})$, $f \not\equiv 0$, *and let* $(\lambda_n)_{n\geq 1}$ *be the zero sequence of* f *in* $\mathbb{D}$, *where each zero is repeated according to its multiplicity. Suppose that*

$$\liminf_{r\to 1} \int_{\mathbb{T}} \log |f_r| dm < \infty,$$

then $\sum_{n\geq 1}(1-|\lambda_n|) < \infty$. *In particular, this holds whenever* $f \in H^p(\mathbb{D})$, $p > 0$.

Remark 5.19. The condition $\sum\limits_{n\geq 1}(1-|\lambda_n|) < \infty$ is called Blaschke condition.

Proof. Without loss of generality, we can assume that $f(0) \neq 0$. But then Jensen's formula gives

$$\sum_{n\geq 1} \log \frac{1}{|\lambda_n|} = \liminf_{r\to 1} \sum_{|\lambda_n|\leq r} \log \frac{r}{|\lambda_n|} < \infty$$

As $|\lambda_n| \to 1$, we have $\log\left(\frac{1}{|\lambda_n|}\right) \sim (1-|\lambda_n|)$, and hence the desired conclusion followed. The $H^p(\mathbb{D})$ case is a consequence of the obvious estimate $\log x < C_p x^p$ for $x > 0$, $p > 0$, because

$$\liminf_{r\to 1} \int_{\mathbb{T}} \log |f_r| \leq \liminf_{r\to 1} \int_{\mathbb{T}} C_p |f_r|^p < \infty.$$

$\square$

For $\lambda \in \mathbb{D}$, we define Blaschke factor by

$$b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{(\lambda - z)}{(1-\bar{\lambda} z)}.$$

(i) If we assume the normalization $b_\lambda\Big(-\frac{\lambda}{|\lambda|}\Big) = 1$, then for $\lambda = 0$, we can define $b_0(z) = z$.

(ii) Zero set $Z(b_\lambda) = \{\lambda\}$, $b_\lambda \in \mathrm{Hol}(\mathbb{C} \setminus \{\frac{1}{\bar\lambda}\})$, $|b_\lambda| \le 1$ on $\mathbb{D}$ and $|b_\lambda| = 1$ on $\mathbb{T}$.

Lemma 5.20 (Blaschke, 1915)**.** *If $(\lambda_n)_{n\ge1} \in \mathbb{D}$ satisfies the Blaschke condition $\sum_{n\ge1}(1 - |\lambda_n|) < \infty$, then the infinite product*

$$B = \prod_{n\ge1} b_{\lambda_n} = \lim_{r\to1} \prod_{|\lambda_n|<r} b_{\lambda_n}$$

converges uniformly on compact subsets of $\mathbb{D}$ and even on compact subsets of $\mathbb{C}\setminus\mathrm{clos}\{\frac{1}{\bar\lambda_n}\}_{n\ge1}$. Moreover, $|B| \le 1$ in $\mathbb{D}$, $|B| = 1$ a.e. on $\mathbb{T}$, and $Z(B) = (\lambda_n)_{n\ge1}$ (counting multiplicity).

Proof. Set $B^r = \prod_{|\lambda_n|<r} b_{\lambda_n}$. Then for $0 \le r < R < 1$, we have

$$\begin{aligned} \|B^R - B^r\|_2^2 &= 2 - 2\,\mathrm{Re}(B^R, B^r) \\ &= 2 - 2\,\mathrm{Re}\int B^R \bar{B}^r dm \\ &= 2 - 2\,\mathrm{Re}\int \frac{B^R}{B^r} dm \text{ (because } |B^r| = 1 \text{ on } \mathbb{T}). \end{aligned}$$

So by MVT for holomorphic function $\frac{B^R}{B^r}$ we get

$$\|B^R - B^r\|_2^2 = 2 - 2\,\mathrm{Re}\left(\frac{B^R}{B^r}\right)(0) = 2 - 2 \prod_{r\le|\lambda_n|<R} |\lambda_n|.$$

By Blaschke condition $\sum_{n\ge1} \log|\lambda_n|^{-1} < \infty$, the product

$$\prod_{n\ge1} |\lambda_n|$$

converges, which implies $\lim_{r\to1} \prod_{r\le|\lambda_n|<R} |\lambda_n| = 1$. This shows that (B^r) is a Cauchy sequence in $H^2 \subset L^2$ for every $r = r_k \to 1$. Hence we deduce the existence of $B = \lim_{r\to1} B^r$. Moreover, $|B| = 1$ a.e. on $\mathbb{T}$ because $|B^r| = 1$ on $\mathbb{T}$, and $B \in H^2$. As the point evaluation is continuous linear functional on H^2, the limit $\lim_{r\to1} B^r(\lambda) = B(\lambda)$ exists uniformly on compact subsets of $\mathbb{D}$, and hence $|B(\lambda)| \le 1$, $\lambda \in \mathbb{D}$. Using $\frac{B}{B^r} \to 1$ in H^2 (this follows by a routine approximation argument), we get $\frac{B}{B^r} \to 1$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$ and

$$\lim_{r\to1} \left(\frac{B}{B^r}\right)(\lambda) = 1. \tag{5.5.1}$$

This shows that $B(\lambda) = 0$, $|\lambda| < 1$ if and only if $\lambda = \lambda_n$ for some $n \ge 1$ (counting multiplicity). If $\lambda \neq \lambda_n$ and $B(\lambda) = 0$, then (5.5.1) will fail.

In order to prove convergence on compact subsets of $\mathbb{C} \setminus \operatorname{clos}\{\frac{1}{\bar{\lambda}_n}\}_{n\geq 1}$, the following observation is enough.

$$|b_{\lambda_n} - 1| = \frac{(|1-|\lambda_n|)(\lambda_n + |\lambda_n|z)}{\lambda(1-\bar{\lambda}z)} \leq \frac{(1-|\lambda_n|)(1+|z|)}{|\lambda_n|\left|z - \frac{1}{\bar{\lambda}_n}\right|} \leq c\frac{1-|\lambda|}{\operatorname{dist}(z,N)},$$

where $N = \operatorname{clos}\{\frac{1}{\bar{\lambda}_n} : n \geq 1\}$. □

Corollary 5.21 (Frigyes Riesz, 1923)**.** *Let $f \in H^p(\mathbb{D})$, $p > 0$ with corresponding zero sequence $(\lambda_n)_{n\geq 1}$. Then there exists $g \in H^p(\mathbb{D})$ with $g(\xi) \neq 0$, $\forall\, \xi \in \mathbb{D}$ such that $f = Bg$ and $\|f\|_p = \|g\|_p$ on $L^p(\mathbb{T})$.*

This may be thought as the Blaschke filtering of the holomorphic functions.

Proof. Take $B^r = \prod_{|\lambda_n|<r} b_{\lambda_n}$, $0 < r < 1$. In particular, $\frac{f}{B^r} \in \operatorname{Hol}(\mathbb{D})$, and for $\rho \to 1$, we get $|B^r(\rho\xi)| \to 1$ uniformly on $\mathbb{T}$. Hence,

$$\left\|\frac{f}{B^r}\right\|_p^p = \lim_{\rho\to 1}\int_{\mathbb{T}}\left|\frac{f}{B^r}(\rho\xi)\right|^p dm(\xi) = \|f\|_p^p \tag{5.5.2}$$

And thus by definition of $H^p(\mathbb{D})$,

$$\left(\int_{\mathbb{T}}\left|\frac{f}{B^r}(\rho\xi)\right|^p dm(\xi)\right)^{\frac{1}{p}} \leq \|f\|_p \text{ for every } 0 \leq \rho < 1.$$

Fix ρ, set $g = \frac{f}{B}$, and letting $r \to 1$, we obtain

$$\left(\int_{\mathbb{T}}\left|g(\rho\xi)\right|^p dm(\xi)\right)^{\frac{1}{p}} \leq \|f\|_p,$$

and hence $\|g\|_p \leq \|f\|_p$. The other inequality follows from $g = \frac{f}{B}$. □

Note. In the proof of (5.5.2) we use the following elementary fact: if $f_\rho \to f$ in the H^p norm and $g_\rho \to 1$ uniformly as $\rho \to 1$, then $f_\rho g_\rho \to f$ in the H^p norm. Indeed,

$$|f_\rho g_\rho - f| \leq |f_\rho g_\rho - f_\rho| + |f_\rho - f|,$$

and one then applies Minkowski's inequality together with the dominated convergence theorem, using that the family (g_ρ) is uniformly bounded by some constant M.

Question 5.22. * Is it possible to replace $\log|\cdot|$ in Jensen's inequality with some suitable increasing function?

Remark 5.23. It is useful to introduce the notion of the zero divisor (or multiplicity function) of a holomorphic function. For $f \in \mathrm{Hol}(\Omega)$, $\Omega \subset \mathbb{C}$, $f \not\equiv 0$, $\lambda \in \Omega$, set

$$d_f(\lambda) = \begin{cases} 0 & \text{if } f(\lambda) \neq 0 \\ m & \text{if } f(\lambda) = \cdots = f^{(m-1)}(\lambda) = 0 \text{ and } f^m(\lambda) \neq 0. \end{cases}$$

The value of $d_f(\lambda)$ is called zero multiplicity of λ. We can redefine the Blaschke condition. The zero divisor of $f \in \mathrm{Hol}(\mathbb{D})$ verifies the Blaschke condition if and only if

$$\sum_{\lambda \in \mathbb{D}} d_f(\lambda)(1 - |\lambda|) < \infty.$$

The corresponding Blaschke product is given by

$$\prod_{\lambda \in \mathbb{D}} b_\lambda^{d_f(\lambda)} = \prod_{n \geq 1} b_{\lambda_n}^{d_f(\lambda_n)}.$$

Corollary 5.24. *Let $f \in H^p, p > 0$ then there exists $f_k \in H^p\,; k = 1,2$ such that $f = f_1 + f_2$, $\|f_k\|_p \leq \|f\|_p$, and $f_k(z) \neq 0$ for $z \in \mathbb{D}$*

Proof. If $f(z) \neq 0$, we may take $f_1 = f_2 = \frac{1}{2}f$. If f has zeros, we have $f = Bg$, with $g \in H^p$ has no zeros. Thus $f(z) = [B(z) - 1]g(z) + g(z)$. □

5.6 Non-tangential boundary limits and Fatou's theorem

Recall that we have identified boundary limit $\tilde{f}$ of $f \in H^p(\mathbb{D})$ via

$$\lim_{r \to 1} \|f_r - \tilde{f}\|_p = 0,\ \tilde{f} \in H^p,\ 1 \leq p < \infty.$$

We shall see another convergence of $f(z)$ to its boundary values, namely the so-called non-tangential convergence a.e. on $\mathbb{T}$ for $f \in H^p(\mathbb{D})$ with $0 < p \leq \infty$.

Let μ be a complex valued Borel measure on $\mathbb{T}$ and $\mu \in \mathcal{M}(\mathbb{T})$. Let $d\mu = hdm + d\mu_s$, $h \in L^1(m)$ be Lebesgue decomposition of μ with respect to m. Then the derivative of μ with respect to m exists at almost every point $\xi \in \mathbb{T}$, in the following sense.

$$\lim_{\Delta \to \xi,\, \xi \in \Delta} \frac{\mu(\Delta)}{m(\Delta)} = \frac{d\mu(\xi)}{dm} (= h(\xi)),$$

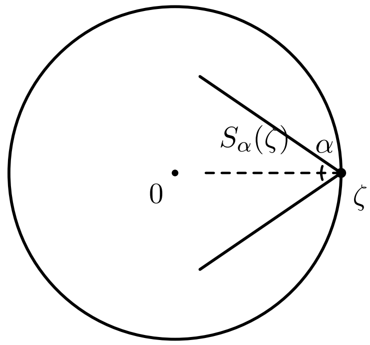

Figure 5.3: A Stolz angle at $\zeta \in \mathbb{T}$ (a typical non-tangential approach region).

where Δ is an arc on $\mathbb{T}$ tending to ξ. Such a point will be called **Lebesgue point** of μ.

Definition 5.25 (Stolz (non-tangential) approach region)**.** Fix $\zeta \in \mathbb{T}$ and $\alpha > 1$. The **Stolz region** (or **non-tangential approach region**) at ζ with aperture α is

$$S_\alpha(\zeta) := \Big\{ z \in \mathbb{D} : \ |z-\zeta| < \alpha\,(1-|z|) \Big\}.$$

A limit of the form $\lim_{z\to\zeta,\ z\in S_\alpha(\zeta)} f(z)$ is called a **non-tangential limit** of f at ζ.

See Figure 5.3 for a schematic illustration.

Since the Poisson kernel satisfies $P(re^{i\theta}) = \dfrac{1-r^2}{|1-re^{i\theta}|^2}$, for $f \in L^p(\mathbb{T})\,(1 \le p < \infty)$, we have

$$\begin{aligned} P_r * f(e^{i\theta}) &= \int_{\mathbb{T}} \frac{1-r^2}{|1-re^{i(\theta-s)}|^2} f(e^{is})\,dm(e^{is}) \\ &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} f(\zeta)\,dm(\zeta), \qquad (z = re^{i\theta},\, \zeta = e^{is}) \\ &=: f * P(z). \end{aligned}$$

Thus $P_r * f(e^{i\theta}) = f * P(z)$, where $z = re^{i\theta} \in \mathbb{D}$; this is the **Poisson integral of** f.

We now come to one of the fundamental results on the non-tangential boundary behaviour of Poisson integrals.

Theorem 5.26 (Fatou)**.** *Let $\mu \in \mathcal{M}(\mathbb{T})$ be a finite (complex) Borel measure, and write its Lebesgue decomposition as*

$$d\mu = h\,dm + d\mu_s, \qquad h \in L^1(\mathbb{T}),\ \mu_s \perp m.$$

Define the Poisson integral (harmonic extension) of μ *by*

$$\mathscr{P}(z) := \int_{\mathbb{T}} P_z(\zeta)\, d\mu(\zeta), \qquad z \in \mathbb{D}.$$

Then for m*-almost every* $\zeta \in \mathbb{T}$ *and for every* $\alpha > 1$*,*

$$\lim_{z\to\zeta,\ z\in S_\alpha(\zeta)} \mathscr{P}(z) = h(\zeta).$$

In particular, if $\mu = h\,dm$ *is absolutely continuous, then* $\mathscr{P}(z) = P[h](z)$ *has non-tangential boundary limits equal to* h *at every Lebesgue point of* h*.*

Proof. We first treat the absolutely continuous case $\mu = h\,dm$ with $h \in L^1(\mathbb{T})$. Fix $\zeta \in \mathbb{T}$ which is a Lebesgue point of h, and fix $\alpha > 1$. For $z \in S_\alpha(\zeta)$, write

$$\mathscr{P}(z) - h(\zeta) = \int_{\mathbb{T}} \Big(h(\eta) - h(\zeta)\Big)\, P_z(\eta)\, dm(\eta).$$

Let $I_z \subset \mathbb{T}$ be the arc centered at ζ of length comparable to $1 - |z|$; more precisely, choose I_z so that $m(I_z) = c_\alpha(1 - |z|)$ and $\eta \in I_z$ iff $|\eta - \zeta| \le C_\alpha(1 - |z|)$. Standard estimates for the Poisson kernel in Stolz regions yield

$$P_z(\eta) \le \frac{C_\alpha}{m(I_z)}\, \mathbf{1}_{I_z}(\eta) + C_\alpha\, \mathbf{1}_{\mathbb{T}\setminus I_z}(\eta), \qquad \eta \in \mathbb{T}, \tag{5.6.1}$$

and moreover $\int_{\mathbb{T}\setminus I_z} P_z\, dm \to 0$ as $z \to \zeta$ within $S_\alpha(\zeta)$. Using (5.6.1) and splitting the integral over I_z and $\mathbb{T} \setminus I_z$, we obtain

$$|\mathscr{P}(z) - h(\zeta)| \le \frac{C_\alpha}{m(I_z)} \int_{I_z} |h(\eta) - h(\zeta)|\, dm(\eta) + \int_{\mathbb{T}\setminus I_z} |h(\eta) - h(\zeta)|\, P_z(\eta)\, dm(\eta).$$

As $z \to \zeta$ non-tangentially, $m(I_z) \to 0$, hence the first term tends to 0 by the Lebesgue differentiation theorem. For the second term, we use that P_z is a probability density and that the mass of P_z escapes from $\mathbb{T} \setminus I_z$ as above; dominated convergence then gives that the second term tends to 0. This proves the conclusion for $\mu = h\,dm$.

For a general measure $\mu = h\,dm + \mu_s$, linearity gives $\mathscr{P} = \mathscr{P}_{ac} + \mathscr{P}_s$ with $\mathscr{P}_{ac} = P[h]$ and $\mathscr{P}_s = P[\mu_s]$. It is a classical consequence of Fatou's theorem (see, e.g., [13]) that the Poisson integral of a singular measure has non-tangential limit 0 for m-almost every boundary point. Hence $\mathscr{P}(z) \to h(\zeta)$ for m-a.e. ζ. □

Corollary 5.27. *If* $f \in H^p(\mathbb{D})$, $0 < p \le \infty$, *then the non-tangential boundary limits of* f

exist a.e. on $\mathbb{T}$*. That is,*

$$\lim_{z\to\xi, z\in S_\xi} f(z) = \tilde{f}(\xi) \ \textit{for a.e. } \xi \in \mathbb{T}.$$

The boundary function $\xi \mapsto \tilde{f}(\xi)$ *is in* $L^p(\mathbb{T})$*, and for* $p \geq 1$*,* $f(\xi) = \tilde{f}(\xi)$ *a.e. on* $\mathbb{T}$ *(*$\tilde{f}$ *is defined in Theorem 5.9).*

Proof. For $p \geq 1$, the claim follows from Fatou's Theorem (5.26) and the Identification Theorem 5.9 (because radial limit exists).

Note that for $f \in L^p(\mathbb{T})$ $(1 \leq p < \infty)$ and $d\mu = f dm$, we have

$$\begin{aligned} P * \mu(z) &= \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} f(\zeta) dm(\zeta) \\ &= P_r * f(\xi) (\text{ let } z = r\xi) \\ &= f_r(\xi) = f_{(r)}(\xi) = f(r\xi) \to \frac{d\mu}{dm}(\xi) = f(\xi) \text{ as } r \to 1 (\text{Fatou's Theorem.}) \end{aligned}$$

Next by identification Theorem 5.9 (i) $f_r \to \tilde{f}$ in L^p, as $r \to 1$. Since convergence in L^p, there exists a subsequence (r_k) such that $P * \mu(\xi) \to \tilde{f}(\xi)$ as $r_k \to 1$ for a.e. $\xi \in \mathbb{T}$ (since convergence in L^p implies there exists a subsequence which is pointwise almost everywhere convergence).

Hence $f(\xi) = \tilde{f}(\xi)$ for a.e. $\xi \in \mathbb{T}$.

For the general case $p > 0$, we know that $f = Bg = B(g^{1/p})^p$, where $g \in H^p(\mathbb{D})$. This implies $g^{1/p} \in H^1(\mathbb{D})$. The result follows from the previous reasoning. □

Notation: From now onward, we identify the functions $f \in H^p(\mathbb{D})$ with their boundary values on $\mathbb{T}$, and write $H^p(\mathbb{D}) = H^p(\mathbb{T})$, $0 < p \leq \infty$, where $H^p(\mathbb{T})$ is the collection of boundary functions of $H^p(\mathbb{D})$.

5.7 The Riesz - Smirnov canonical factorization

Section roadmap.

- Separate the zero set through a Blaschke product.
- Remove the singular part through a singular inner factor detected by the boundary modulus.

- Show that the remaining factor is outer and that the decomposition is unique up to unimodular constants in the expected places.

Here we see the main result of the Hardy space theory - a parametric representation of $f \in H^p$ as a product of Blaschke product, a singular inner function, an outer (maximal) function. The last two functions are exponential of integral depending on the holomorphic Schwarz - Herglotz kernel $z \to \frac{\zeta+z}{\zeta-z}$, whose real part is the Poisson kernel.

Theorem 5.28. *Let $f \in L^p$, $0 < p \leq \infty$ be such that $\log|f| \in L^1$, and define*

$$[f](z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} \log|f(\zeta)| dm(\zeta)\right), \ |z| < 1.$$

Then

(i) $[f] \in H^p(\mathbb{D})$ *and* $|[f]| = |f|$ *a.e. on* $\mathbb{T}$.

(ii) If $0 \not\equiv g \in H^q(\mathbb{D})$, $q \geq 1$, *and* $|g| \leq |f|$ *a.e. on* $\mathbb{T}$, *then* $|g| \leq |[f]|$ *on* $\mathbb{D}$ (*and hence* $g \in H^p(\mathbb{D})$).

(iii) $\left[\frac{f}{g}\right] = \frac{[f]}{[g]}$ *and* $[[f]] = [f]$.

(iv) $[f](z) \neq 0$ *in* $\mathbb{D}$ *and for any* $\alpha > 0$, $[|f|^\alpha] = [f]^\alpha$.

Proof. (i) For fixed z, $|\frac{\zeta+z}{\zeta-z}| < \infty$ and $\log|f| \in L^1$ hence $[f](z)$ is well defined. Consequently, $[f]$ is a holomorphic function on $\mathbb{D}$. Recall that for a finite Borel measure μ and a convex function ψ, we have the Jensen-Young geometric mean inequality

$$\frac{\int \psi \circ F \, d\mu}{\int d\mu} \geq \psi\left(\frac{\int F d\mu}{\int d\mu}\right). \tag{5.7.1}$$

Proof of (5.7.1). Set $\nu = \mu/(\int d\mu)$, so that ν is a probability measure. Since ψ is convex on I, it is the supremum of its affine minorants:

$$\psi(x) = \sup\{h(x) : \ h(x) = ax + b \text{ and } h \leq \psi \text{ on } I\}.$$

For any such h we have $h\Big(\int F \, d\nu\Big) = \int h \circ F \, d\nu \leq \int \psi \circ F \, d\nu$. Taking the supremum over all affine minorants yields (5.7.1).

By applying inequality (5.7.1) to the Borel measure $d\mu = \frac{1-|z|^2}{|\zeta-z|^2}\, dm(\zeta)$, we get (5.7.1) to the Borel measure $d\mu = \frac{1-|z|^2}{|\zeta-z|^2}\, dm(\zeta)$, we get

$$|[f]|^p = \exp\Big(\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2}\log|f(\zeta)|^p dm(\zeta)\Big) \le \int_{\mathbb{T}} |f(\zeta)|^p \frac{1-|z|^2}{|\zeta-z|^2} dm(\zeta).$$

Set $z = re^{it}$. By Fubini's theorem, we get

$$\int_0^{2\pi} \Big|[f](re^{it})\Big|^p \frac{dt}{2\pi} \le \int_{\mathbb{T}} |f(\zeta)|^p \Big(\int_0^{2\pi} \frac{1-|z|^2}{|\zeta-z|^2}\frac{dt}{2\pi}\Big) dm(\zeta) = ||f||_p^p.$$

Next, by Fatou's theorem and its corollary there, we have

$$\log|[f](\xi)| = \lim_{r\to1}\log|[f](r\xi)| = \log|f(\xi)| \text{ a.e. } \xi \text{ on } \mathbb{T}.$$

The modifications in the case $p=\infty$ are obvious.

(ii) Given that $0 \not\equiv g \in H^q(\mathbb{D})$, $q \ge 1$, and $|g| \le |f|$ a.e. on $\mathbb{T}$. This implies $\log|g| \in L^1$, and hence by generalized Jenson's inequality (5.3.4), we get

$$\begin{aligned}
\log|g(z)| &\le \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2}\log|g(\zeta)|dm(\zeta) \\
&\le \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2}\log|f(\zeta)|dm(\zeta) \\
&= \log|[f](z)|.
\end{aligned}$$

(iii) is a direct consequence of the definition.

(iv) It is a direct consequence of the definition. But here we only consider the fact $\log|f|^\alpha \in L^1$, whereas $f^\alpha \in L^p$ is not considered.

□

Note: $^*1\Big[$

$$\frac{\xi+z}{\xi-z} = 1 + 2\sum_{n=1}^{\infty}\frac{z^n}{\xi^n} \quad \text{since} \quad |\frac{z}{\xi}| < 1.$$

Since $z \in \mathbb{D}, z = r\xi, \xi \in \mathbb{T}$

$$\left|\frac{\xi+z}{\xi-z}\right| \le 1 + 2\sum_{n=1}^{\infty} r^n = 1 + 2(\frac{1}{1-r} - 1) = \frac{1+r}{1-r} < \infty$$

Since r fixed for fixed z.]

Observe that Theorem 5.28 requires only the condition $\log|f| \in L^1$ in order to define $[f]$; the additional assumption $f \in L^p$ guarantees that $[f] \in H^p(\mathbb{D})$.

The following result ensures the existence of enough harmonic functions as Poisson integrals of finite Borel measures.

Theorem 5.29. (*G. Herglotz, 1911*) *Let u be a non-negative harmonic function on $\mathbb{D}$. Then there exists a unique finite Borel measure $\mu \geq 0$ such that $u = P * \mu$, that is*

$$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

Proof. By MVT we have for all z in $\mathbb{D}$

$$u_r(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} u_r(\zeta) dm(\zeta) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu_r(\zeta),$$

where we have set $u_r(z) = u(rz)$, $0 \leq r < 1$, and $d\mu_r = u_r dm$. Then μ_r is a positive measure and $\mathrm{Var}(\mu_r) = \mu_r(\mathbb{T}) = u_r(0) = u(0) < \infty$. Thus the family $(u_r)_{0 \leq r < 1}$ is uniformly bounded in $\mathcal{M}(\mathbb{T})$, and has a weak* convergent subsequence μ_{r_n} that converges to $\mu \in \mathcal{M}(\mathbb{T})$. Recall that $\mathcal{M}(\mathbb{T})$ is the dual of $C(\mathbb{T})$, with the pairing $< f, \mu >= \int_{\mathbb{T}} f d\mu$. Thus, if $f \in C(\mathbb{T})$ and $f \geq 0$, then

$$\int_{\mathbb{T}} f d\mu = \lim_{n\to\infty} \int_{\mathbb{T}} f u_{r_n} dm \geq 0 \implies \mu \geq 0.$$

Moreover, since u is continuous on $\mathbb{D}$, for $z \in \mathbb{D}$, we have

$$u(z) = \lim_{n\to\infty} u(r_n z) = \lim_{n\to\infty} \int \frac{1-|z|^2}{|\zeta - z|^2} d\mu_{r_n}(\zeta) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

Uniqueness of μ. Note that $P * \mu(re^{it}) = \sum_{n\in\mathbb{Z}} r^{|n|}\hat{\mu}(n) e^{int}$. For any ν such that $P * \mu = P * \nu$ implies $\hat{\mu}(n) = \hat{\nu}(n)$. Hence $\mu = \nu$. $\square$

Theorem 5.30. (*Singular inner function*)*: Let $S \in \mathrm{Hol}(\mathbb{D})$, then the following are equivalent:*

(i) $|S(z)| \leq 1$ *and* $S(z) \neq 0$ *on* $\mathbb{D}$, $S(0) > 0$ *and* $|S(\xi)| = 1$ *a.e. on* $\mathbb{T}$.

(ii) there exists a unique finite Borel measure $\mu \geq 0$ on $\mathbb{T}$ with $\mu \perp m$ such that

$$S(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right),\ z \in \mathbb{D}.$$

Proof. ($\Longleftarrow$) Assume (ii). Then $S \in H^\infty(\mathbb{D})$ and

$$|S(z)| = \exp\left(-\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(\zeta)\right), \qquad z \in \mathbb{D}.$$

Since $\mu \perp m$, Fatou's theorem applied to the harmonic function $u(z) := -\log|S(z)|$ yields

$$\lim_{r\to 1} u(r\xi) = \frac{d\mu}{dm}(\xi) = 0 \qquad \text{for a.e. } \xi \in \mathbb{T}.$$

Hence $|S(r\xi)| \to 1$ for almost every $\xi \in \mathbb{T}$. As already noted, $|S(z)| \leq 1$ on $\mathbb{D}$, S has no zeros in $\mathbb{D}$, and $S(0) > 0$ by construction. Thus (i) holds.

($\Longrightarrow$) Assume (i), and set $u = \log|S|^{-1}$. Then u is a non-negative harmonic function on $\mathbb{D}$, so by the Herglotz theorem there exists a finite positive Borel measure μ on $\mathbb{T}$ such that

$$\log|S(z)|^{-1} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(\zeta).$$

Since $|S(\xi)| = 1$ almost everywhere on $\mathbb{T}$, another application of Fatou's theorem gives

$$\frac{d\mu}{dm}(\xi) = \lim_{r\to 1} u(r\xi) = 0 \qquad \text{for a.e. } \xi \in \mathbb{T}.$$

Therefore $\mu \perp m$.

Let S_μ be the function defined from this measure by the formula in (ii). Then $|S| = |S_\mu|$ on $\mathbb{D}$, so S/S_μ is a holomorphic function of constant modulus 1. Hence $S = \lambda S_\mu$ for some unimodular constant λ. Since both $S(0)$ and $S_\mu(0)$ are positive, we must have $\lambda = 1$. Thus $S = S_\mu$. □

Definition 5.31. A nonconstant inner function with no zeros in $\mathbb{D}$ is called a **singular inner function**. Equivalently, a function S satisfying either of the conditions in the preceding theorem is a singular inner function. The term "singular" refers to its representation in terms of a measure singular with respect to Lebesgue measure.

Notation 5.32. $\log^+ x = \begin{cases} \log x, x \geq 1 \\ 0, 0 < x < 1 \end{cases}$ and $\log^- x = \begin{cases} -\log x, 0 < x \leq 1 \\ 0, x > 1 \end{cases}$

Then $\log x = \log^+ x - \log^- x$ and $|\log x| = \log^+ x + \log^- x$ for $x > 0$. Moreover, $\log^+ x \leq x$ for $x > 0$, and $|\log^+ x - \log^+ y| \leq |x - y|$ for all $x, y > 0$.

Theorem 5.33 (Smirnov, 1928: Canonical Factorization Theorem)**.** *Let $f \in H^p(\mathbb{D})$, $p > 0$. Then there exists a unique factorization $f = \lambda BS[f]$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1, B$, S and $[f]$ are defined earlier.*

Proof. First set

$$g = \tfrac{f}{B}.$$

We will show that any zero free function g satisfies $\int_{\mathbb{T}} \log|g|dm > -\infty$. We may assume $g(0) = 1$. Since g has no zeros in $\mathbb{D}$, $\log|g(z)|$ is harmonic in $\mathbb{D}$. The MVT for the harmonic function says that any for any $r \in (0,1)$

$$\begin{aligned} 0 = \log|g(0)| &= \int_{\mathbb{T}} \log|g(r\xi)|dm(\xi) \\ &= \int_{\mathbb{T}} \log^+|g(r\xi)|dm(\xi) - \int_{\mathbb{T}} \log^-|g(r\xi)|dm(\xi) \end{aligned}$$

Thus $\int_{\mathbb{T}} \log^+|g(r\xi)|dm(\xi) = \int_{\mathbb{T}} \log^-|g(r\xi)|dm(\xi) \leq \int_{\mathbb{T}} |g(r\xi)|dm(\xi)dm(\xi) \leq \|g\|$ (Cauchy Schwartz). Since $g \in H^p(\mathbb{D})$, g along with the functions $\log^+|g|$ and $\log^-|g|$ have radial limits a.e. on $\mathbb{T}$. By Fatou's lemma

$$\int_{\mathbb{T}} \log^-|g|dm \leq \lim_{r\to 1} \int_{\mathbb{T}} \log^-|g(r\xi)|dm(\xi) \leq \|g\|$$

which implies that $\log^{-1}|g|$ is integrable on $\mathbb{T}$. Simillary $\log^+|g|$ and $\log g$ is integrable.

Then $|f| = |g|$ a.e. on $\mathbb{T}$, and hence $[g] = [f]$. Set $\lambda = \frac{g(0)}{[g](0)}$ and $S = \frac{g}{\lambda[g]}$. Then $f = Bg = B\lambda S[g] = \lambda BS[f]$. As B and $[f]$ are uniquely defined for f, the uniqueness of factorization follows. $\square$

Next, we consider the structure of the outer functions in H^p.

Theorem 5.34. (***Structure of outer function***) *Let $p, q, r \geq 1$ and $f \in H^p$. Then the following are equivalent.*

(i) There exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $f = \lambda[f]$.

(ii) for all $z \in \mathbb{D}$, the generalized Jensen inequality is equality:

$$\log|f(z)| = \int_{\mathbb{T}} P(z\bar{\xi}) \log|f(\xi)|dm(\xi). \tag{5.7.2}$$

(iii) Identity (5.7.2) holds for at least one $z \in \mathbb{D}$.

(iv) If $g \in H^q$ and $\frac{g}{f} \in L^r$, then $\frac{g}{f} \in H^r$ (Integral Maximal principle).

If $p = 2$, then (i)-(iv) are equivalent to

(v) the function f is outer in H^2(In the earlier sense i.e., $E_f = H^2$).

Proof. (i) implies (ii) is followed from the definition of $[f]$. The implication (iii) goes to (ii) is trivial. For (iii) implies (i), suppose (5.7.2) holds for some $z_o \in \mathbb{D}$. By Riesz-Smirnov factorization theorem, we have $f = \lambda BS[f]$, and by (5.7.2), we get

$$|f(z_o)| = |\lambda B(z_o)S(z_o)[f](z_o)| \implies |B(z_o)S(z_o)| = 1 \implies |B(z_o)| = |S(z_o)| = 1.$$

By maximum principle, $B = S =$constant$= 1$ in $\mathbb{D}$, implies $f = \lambda[f]$.
(i) implies (iv): If $g \in H^q$, then $g = \lambda_1 BS[g]$ and we get $\frac{g}{f} = \frac{\lambda_1 BS[g]}{(\lambda[f])} = \left(\frac{\lambda_1}{\lambda}\right) BS\left[\frac{g}{f}\right] \in H^r$ in view of Riesz-Smirnov theorem and by the hypothesis that $g/f \in L^r$.
(iv) $\implies$ (i): Let $f = \lambda BS[f]$ and set $g = \min(|f|, 1)$. Then $[g] \in H^\infty$ and $\left|\frac{[g]}{f}\right| \leq 1$ a.e. on $\mathbb{T}$. By (iv) we get $\frac{[g]}{f} \in H^r$ (r arbitrary). Again, we have $\frac{[g]}{f} = \lambda_1 B_1 S_1 \left[\frac{g}{f}\right] = \lambda_1 B_1 S_1 \frac{[g]}{[f]}$ (because $[[g]] = [g]$ and $\left[\frac{g}{f}\right] = \frac{[g]}{[f]}$), we get $1 \equiv \lambda\lambda_1 BB_1 SS_1 = \lambda_2 B_2 S_2$ with $|\lambda_2| = 1$, where B_2 is a Blaschke product and S_2 is a singular inner function. As $|B_2(z)| \leq 1$ and $|S_2(z)| \leq 1$ for all $z \in \mathbb{D}$, we get $|B_2| = |S_2| \equiv 1$ and hence $B_2 \equiv S_2 \equiv 1$. Thus, we conclude that $B = S = 1$, implies $f = \lambda[f]$.

It remains to show that (iv) and (v) are equivalent if $p = 2$. As (i)-(iii) are independent of choice of q and r, we get equivalence between (iv) as well with $p = 2$, and arbitrary q, r and with $p = q = r = 2$, (iv) is just earlier characterization of the outer function on H^2. ☐

Remark 5.35. In the family of Hardy spaces, dividing by an analytic function, even if it does not have any zero, is a delicate process and the result could be a function that does not belong to any Hardy space. For example, if S is a singular inner function, then $1/S$ does not belong to any Hardy space (as one readily verifies). However, at the same time, its boundary values are unimodular and one is (wrongly) tempted to say that $1/S$ is an inner function. The above result (see Theorem 5.34(iv)) says that dividing by an outer function is legitimate as long as the boundary values remain in a Lebesgue space.

Definition 5.36 (Outer in H^p)**.** Let $f \in H^p$, $p > 0$ and $f = \lambda BS[f]$. The function $[f]$ is called the outer part of f, and λBS is called the inner part of f. We write $[f] = f_{out}$ and $\lambda BS = f_{inn}$. If $f = \lambda[f]$, then f is called **outer**.

It is clear from the above theorem that if $p = 2$, then definition of inner and outer functions coincide with previous ones.

Corollary 5.37. *Let $w \in L^1_+(\mathbb{T})$, and $p \geq 1$. The following are equivalent.*

(i) There exists $f \in H^p$, $f \not\equiv 0$ such that $|f|^p = w$ a.e. on $\mathbb{T}$.

(ii) $\log w \in L^1$.

Proof. As $H^p \subset H^1$, and $p \geq 1$ (i) implies (ii) follows from the boundary uniqueness theorem Corollary 5.15.

Next (ii) implies (i) follows by taking $f = [w^{1/p}]$. Since if

$$f(z) := [w^{1/p}](z) = \exp\left(\int_{\mathbb{T}} P(z\bar{\xi}) \log |w(\xi)|^{1/p} dm(\xi)\right),$$

then by Theorem 5.28 (i), $f \in H^p(\mathbb{D})$.

Since

$$|f(z)|^p = \exp\left(\int_{\mathbb{T}} P(z\bar{\xi}) \log |w(\xi)| dm(\xi)\right)$$

by Fatou's theorem 5.26, we get $|f|^p = w$ a.e. on $\mathbb{T}$. □

5.8 Approximation by inner functions and Blaschke products

Using Fatou's theorem, we prove two important theorems on uniform approximation by inner functions.

Theorem 5.38. (*R. Douglas and W. Rudin, 1969*) *Let* Σ *be the set of all inner functions. Then*

$$L^\infty(\mathbb{T}) = \operatorname*{clos}_{L^\infty}\left(\bar{\theta} H^\infty : \theta \in \Sigma\right) = \overline{\operatorname{span}}_{L^\infty}\left(\bar{\theta}_1\theta_2 : \theta_1, \theta_2 \in \Sigma\right). \tag{5.8.1}$$

Moreover, any unimodular function in $L^\infty(\mathbb{T})$ *belongs to*

$$\operatorname*{clos}_{L^\infty}(\Pi)\left(\bar{\theta}_1\theta_2 : \theta_1, \theta_2 \in \Sigma\right).$$

Proof. It is enough to show that $\chi_\sigma \in \overline{\operatorname{span}}_{L^\infty}\left(\bar{\theta}_1\theta_2 : \theta_1, \theta_2 \in \Sigma\right)$ for every Borel measurable set σ in $\mathbb{T}$. Let

$$f_n = \left[n\chi_\sigma + \frac{1}{n}\chi_{\mathbb{T}\setminus\sigma}\right],\ n = 2, 3, \ldots$$

and $A_n = \{z \in \mathbb{C} : \frac{1}{n} < |z| < n\}$. It follows that $f_n(\mathbb{D}) \subset A_n$ (by maximum principle) and $f_n(\mathbb{T}) \subset \partial A_n$. Next let $\phi_1(\zeta) = \zeta + \frac{1}{\zeta}$ for $\zeta \in \mathbb{C} \setminus \{0\}$, and $w : \phi_1(A_n) \to \mathbb{D}$ be a conformal (Riemann) mapping of the ellipse $\phi_1(A_n)$ onto $\mathbb{D}$. Since the boundary of ellipse is smooth, w can be continuously extended to clos $\phi_1(A_n)$, and hence

$$w \circ \phi_1 \circ f_n = \theta_1$$

is an inner function (because $\theta_1 \in H^\infty(\mathbb{D})$, and by Fatou's theorem $|\theta_1| = 1$ a.e. on $\mathbb{T}$). Since w^{-1} is continuous on clos($\mathbb{D}$), it can be approximated by its Fejer polynomials. Therefore,

$$f_n + \frac{1}{f_n} = \phi_1 \circ f_n = w^{-1} \circ \theta_1 \in \operatorname*{span}_{L^\infty} (\theta_1^n : n \geq 0).$$

Doing the same for the function $\phi_2(\zeta) = \zeta - \frac{1}{\zeta}$, we get an inner function θ_2 such that $f_n - \frac{1}{f_n} \in \overline{\text{span}}_{L^\infty} (\theta_2^n : n \geq 0)$. Hence $f_n \in \overline{\text{span}}_{L^\infty}\{\theta_1^k\theta_2^n : k, n \geq 0\}$, implies

$$|f_n|^2 \in \overline{\text{span}}_{L^\infty} \left(\theta_1^k \theta_2^n \theta_1^{-l} \theta_2^{-m} : k, n, l, m \geq 0\right).$$

Thus,

$$\chi_\sigma + \frac{1}{n^4}\chi_{\mathbb{T}\setminus\sigma} \in \overline{\text{span}}_{L^\infty} \left(\bar{\theta}_1\theta_2 : \theta_1, \theta_2 \in \Sigma\right), \text{ for} n = 1, 2, \ldots.$$

Letting $n \to \infty$, we get $\chi_\sigma \in \overline{\text{span}}_{L^\infty} \left(\bar{\theta}_1\theta_2 : \theta_1,\, \theta_2 \in \Sigma\right)$.

Let $u \in L^\infty(\mathbb{T})$, and $|u| = 1$ a.e. and $u_1 \in L^\infty(\mathbb{T})$ with $|u_1| = 1$ a.e. and $u = u_1^2$. Given $\epsilon > 0$, by (5.8.1) there exists $\varphi, \theta_j \in \Sigma$ such that $|u_1 - \bar{\varphi}g| < \epsilon$, where $g = \sum_{j=1}^{n} a_j\theta_j$, $a_j \in \mathbb{C}$. Set $\theta = \prod_{j=1}^n \theta_j$, and observe that $\bar{g}\theta \in H^\infty$. Since $[\bar{g}\theta] = [g]$ (because $|\bar{g}\theta| = |\bar{g}|$), the inner-outer factorizations of g and $\bar{g}\theta$ are of the form $\bar{g}\theta = v[g]$ and $g = w[g]$, where $v, w \in \Sigma$, and $1 - \epsilon < |[g]| < 1 + \epsilon$. Next, $|\bar{u_1} - \varphi\bar{g}| = |\bar{u_1} - \varphi\bar{\theta}v[g]| < \epsilon$ gives

$$\left|\frac{1}{\bar{u_1}} - \frac{1}{\phi\bar{\theta}v[g]}\right| < \frac{\epsilon}{1-\epsilon}.$$

Since $|u_1 - a| < \epsilon$ and $|u_1 - b| < \epsilon$ implies that $|u_1^2 - ab| \leq |u_1 - a| + |a||u_1 - b|$, we obtain

$$\left|u - \bar{\phi}w[g]\bar{\phi}\theta\bar{v}\frac{1}{[g]}\right| < \frac{2\epsilon}{1-\epsilon},$$

which completes the proof. □

Theorem 5.39. (*O. Frostman, 1935*) *Let θ be a (non-constant) inner function and $\zeta \in \mathbb{T}$. Then $b_{t\zeta} \circ \theta$ are Blaschke products with simple zeros for a.e. $t \in (0, 1)$, where $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$, $\lambda \in \mathbb{D}$. In particular, θ is a uniform limit of Blaschke products with simple zeros.*

Proof. Let $\zeta = 1$. Then we need to show that $H_t(z) := b_t \circ \theta(z) = \frac{t - \theta(z)}{1 - t\theta(z)}, z \in \mathbb{D}$ is Blaschke product with simple zeros for all $t \in [0, 1)$. Let $\xi \in \mathbb{T}$, then the boundary function $|\tilde{H}_t(\xi)| = \left|\frac{t - \tilde{\theta}(\xi)}{1 - t\tilde{\theta}(\xi)}\right| = \left|\frac{t - \tilde{\theta}(\xi)}{\overline{\tilde{\theta}(\xi)} - t}\right| = \left|\frac{t - \tilde{\theta}(\xi)}{\overline{t - \tilde{\theta}(\xi)}}\right| = 1 \implies \tilde{H}_t \in H^\infty(\mathbb{T})$. Hence $H_t \in H^\infty(\mathbb{D})$.

By the unique canonical factorization of $H_t(z)$, $H_t(z) = \lambda BS[H_t](z)$ where

$$[H_t](z) = \exp \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log |\tilde{H}_t(\xi)| dm(\xi) = \exp(0) = 1$$

since $|\tilde{H}_t(\xi)| = 1$. Hence $H_t(z) = \lambda BS$. Our claim is to show: $S = 1$, where

$$S(z) = \exp\left(- \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu_t(\xi) \right), \mu_t \perp m, \mu_t \geq 0.$$

To show $S = 1$ we will show $\mu_t(\mathbb{T}) = 0$.

Then, by Jensen's formula (5.12) applied to the singular inner factor S (recall that $S \in H^\infty$ and $\|S\|_\infty \leq 1$), and using the pointwise estimate $|H_t(r\xi)| \leq |S(r\xi)|$ for $0 \leq r < 1$ and $\xi \in \mathbb{T}$, we obtain

$$\mu_t(\mathbb{T}) = \log |S(0)|^{-1} = \int_{\mathbb{T}} \log |S(r\xi)|^{-1} \, dm(\xi) \leq \int_{\mathbb{T}} \log |\tilde{H}_t(r\xi)|^{-1} \, dm(\xi) =: g(r,t),$$

for all $t \in (0,1)$ and $0 \leq r < 1$. In particular, since $g(r,t) \geq 0$, it suffices to prove that $g(r,t) = 0$ for almost every t.

Fix $0 \leq r < 1$ and define

$$u(w) := \int_0^1 \log |b_t(w)|^{-1} \, dt, \qquad w \in \overline{\mathbb{D}}.$$

For each fixed $t \in (0,1)$ the function $w \mapsto \log |b_t(w)|^{-1}$ is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Moreover, if $|w| = 1$ then $|b_t(w)| = 1$, hence $u(w) = 0$ on $\mathbb{T}$. Consequently u is harmonic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and vanishes on the boundary; by the maximum principle, $u \equiv 0$ on $\mathbb{D}$.

Since $H_t = b_t \circ \theta$, we may apply Fubini's theorem to obtain

$$\int_0^1 g(r,t) \, dt = \int_{\mathbb{T}} \int_0^1 \log |H_t(r\xi)|^{-1} \, dt \, dm(\xi) = \int_{\mathbb{T}} u(\theta(r\xi)) \, dm(\xi) = 0.$$

Because $g(r,t) \geq 0$, this implies $g(r,t) = 0$ for almost every $t \in (0,1)$. Returning to the inequality $\mu_t(\mathbb{T}) \leq g(r,t)$, we conclude that $\mu_t(\mathbb{T}) = 0$ for almost every $t \in (0,1)$, hence $S \equiv 1$ for such t.

It remains to note that the zeros are simple for almost every parameter. Indeed, if $b_\lambda(\theta(z_0)) = 0$, then $\theta(z_0) = \lambda$ and

$$(b_\lambda \circ \theta)'(z_0) = b'_\lambda(\theta(z_0)) \, \theta'(z_0) = b'_\lambda(\lambda) \, \theta'(z_0).$$

Thus z_0 is a simple zero provided $\theta'(z_0) \neq 0$. Equivalently, all zeros are simple whenever $\lambda \neq \theta(z_j)$ for every critical point z_j of θ (i.e. for every zero of θ').

Finally, we show that u is continuous on $\overline{\mathbb{D}}$. The integrals $\int_0^1 \log|1 - tw|\,dt$ and $\int_0^1 \log|t-w|\,dt$ are of the same type. Writing $w = x+iy$, we obtain

$$\int_0^1 \log|t-w|^2\,dt = \int_0^1 \log\Big((t-x)^2 + y^2\Big)\,dt,$$

and the right-hand side is continuous in (x,y); for example, $\int_0^1 \log(t-x)^2\,dt = \chi_{(0,1)} * \log(x^2)$. $\square$

5.9 Exercises and further directions

These short problems consolidate the factorization machinery developed in the chapter and emphasize the rigidity of analytic boundary behaviour.

Core exercises

Exercise 5.40. Show that $H^2(\mathbb{D})H^2(\mathbb{D}) = H^1(\mathbb{D})$.

Proof. If $f, g \in H^2(\mathbb{D})$, then by Hölder's inequality,

$$\|fg\|_{H^1} \le \|f\|_{H^2}\,\|g\|_{H^2} < \infty,$$

so $H^2(\mathbb{D})\cdot H^2(\mathbb{D}) \subseteq H^1(\mathbb{D})$.

Conversely, let $F \in H^1(\mathbb{D})$ and write its canonical factorization $F = I\,O$, where I is inner and O is outer. Since $\log|O| \in L^1(\mathbb{T})$, the function $O^{1/2}$ defined by

$$O^{1/2}(z) := \exp\Big(\frac{1}{2}\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\log|\tilde{O}(\zeta)|\,dm(\zeta)\Big)$$

is outer and belongs to $H^2(\mathbb{D})$. Setting $f := O^{1/2}$ and $g := I\,O^{1/2}$, we have $f, g \in H^2(\mathbb{D})$ and $fg = F$. Hence $H^1(\mathbb{D}) \subseteq H^2(\mathbb{D})\cdot H^2(\mathbb{D})$. $\square$

Exercise 5.41. $f \in H^1, f(\mathbb{T}) \subseteq \mathbb{R}$ then f is a constant.

Proof. Let $f = u + iv$ on $\mathbb{D}$, where $u = \Re f$ and $v = \Im f$ are harmonic. Since $f \in H^1(\mathbb{D})$, the non-tangential boundary values $\tilde{f}$ exist almost everywhere on $\mathbb{T}$ and belong to $L^1(\mathbb{T})$. By assumption $\tilde{f}$ is real-valued almost everywhere, hence $\tilde{v} = 0$ almost everywhere on $\mathbb{T}$.

The harmonic function v belongs to the harmonic Hardy class h^1, so it admits the Poisson representation

$$v(re^{it}) = \int_{\mathbb{T}} P_r(e^{it}\bar{\zeta})\, \tilde{v}(\zeta)\, dm(\zeta), \qquad 0 < r < 1,$$

where P_r is the Poisson kernel. Since $\tilde{v} = 0$ almost everywhere, the right-hand side vanishes identically, and therefore $v \equiv 0$ on $\mathbb{D}$. Thus $f(\mathbb{D}) \subseteq \mathbb{R}$. An analytic function with real values on a domain is constant, so f is constant. □

Exercise 5.42. Let $f \in H^{1/2}$. Assume that $f \geq 0$ a.e. on $\mathbb{T}$. Then f is a constant.

Proof. Assume $f \not\equiv 0$. By the canonical factorization theorem, we may write $f = Bg$, where B is the Blaschke product determined by the zeros of f and $g \in H^{1/2}$ is holomorphic and nonvanishing on $\mathbb{D}$. Since $\mathbb{D}$ is simply connected, the nonvanishing of g implies the existence of a holomorphic square root: there is $h \in \mathrm{Hol}(\mathbb{D})$ such that $h^2 = g$. Moreover, the boundary function satisfies $|h| = |g|^{1/2}$ a.e. on $\mathbb{T}$, hence $h \in H^1$. Consequently, $f = Bh^2$.

The condition $f \geq 0$ ensures that $f = |f|$ a.e. on $\mathbb{T}$. Hence, since B is unimodular on $\mathbb{T}$, we have $Bh^2 = \bar{h}$ a.e. on $\mathbb{T}$.

Next on one hand we have, $Bh \in H^1$, and on the other hand $\bar{h} \in \bar{H}^1$. We know that $H^1 \cap \overline{H}^1$ contains only the constant functions. Therefore Bh is a constant function. By the uniqueness of the canonical factorization this happens precisely when B is a unimodular constant and h is a constant. Thus eventually h is a constant. □

Example 5.43. If $f(z) = \exp(\frac{z+1}{z-1})$ then f is a singular inner function.

Proof. Recall that $|e^w| = |e^{\mathrm{Re}\, w + i\, \mathrm{Im}\, w}| = |e^{\mathrm{Re}\, w}| = e^{\mathrm{Re}\, w}$. Hence $|f(z)| = \exp\left(\mathrm{Re}(\frac{z+1}{z-1})\right) = \frac{|z|^2-1}{|z-1|^2} < 0$ for $z \in \mathbb{D}$. It follows that $|f(z)| < 1 \forall z \in \mathbb{D}$. Thus $f \in H^\infty$. Moreover for $|z| = 1$ and $z \neq 1$ implies $\mathrm{Re}\, \frac{z+1}{z-1} = 0$ and therefore $|\tilde{f}(e^{i\theta})| = 1$ for all $\theta \neq 0$. Since e^w is never zero for any complex number w, it follows that f is an inner function with no zeros on $\mathbb{D}$. □

Remark 5.44. The function $f(z) = \exp(\frac{1+z}{1-z})$ is not an inner. This function is the reciprocal of the function in earlier example hence $|f(e^{i\theta})| = 1$ for $\theta \in (0, 2\pi)$. However for $0 < r < 1$

$$|f(r)| = \exp\left(\frac{1+r}{1-r}\right) \to \infty, \quad r \to 1^-$$

Although f has unimodular boundary value almost everywhere on $\mathbb{T}$, it is unbounded on $\mathbb{D}$ and hence is not an inner function. Thus when checking to see whether or not an analytic function is inner one must be careful to check at first that it is actually bounded on $\mathbb{D}$.

Exercise 5.45. Let $r > 0, s > 0, t > 0$ be such that $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$. Show that $H^r = H^s \cdot H^t$ and moreover $\|f_r\| = \min\{\|g\|_s \|h\|_t : g \in H^s, h \in H^t \text{ s.t. } f = gh\}$

Proof. By Hölder's inequality, if $g \in H^s(\mathbb{D}), h \in H^1(\mathbb{D})$ then $f = gh \in \text{Hol}(\mathbb{D})$ and for every $\rho, 0 < \rho < 1$, we have $\|f_\rho\| \le \|g_\rho\|_s \|h_\rho\|_t$, which implies $f \in H^r(\mathbb{D})$ and $\|f\|_r \le \|g\|_s \|h\|_t$. Conversely, if $f \in H^r(\mathbb{D})$, with $f = \lambda BV[f]$ its Canonical factorization, then by $g = \lambda BV[f]^{r/s}, h = [f]^{r/t}$, we obtain $f = gh$ and $\|f\|_r = \|g\|_s \|h\|_t$. □

Exercise 5.46. Let $\lambda \in \mathbb{D}$ and φ_λ be an evaluating functional on H^p, $1 \le p \le \infty$, i.e.

$$\varphi_\lambda(f) = f(\lambda), f \in H^p.$$

Show that $\|\varphi_\lambda\| = (1 - |\lambda|^2)^{-1/p}$.

Proof. When $p = 2, \varphi_\lambda(f) = f(\lambda) = \sum_{k \ge 0} \widehat{f}(k)\lambda^k = (f, k_\lambda)_{H^2}$, where

$$k_\lambda(z) = \sum_{k \ge 0} \overline{\lambda}^k z^k, z \in \mathbb{D},$$

is the Szego reproducing kernel of H^2, hence $\|\varphi_\lambda\| = \|k_\lambda\|_2 = (1 - |\lambda|^2)^{-1/2}$. When p-is arbitrary, recall that for every function f, $|f(\lambda)| \le |[f]\lambda|$ and $\|f\|_p = \|[f]^{p/2}\|_2^{2/p}$ which leads to:

$$\begin{aligned} \|\varphi_\lambda\| = \sup\{|f(\lambda)| : f \in H^p, \|f\|_p \le 1\} &= \sup\{|[f]^{p/2}(\lambda)|^{2/p} : \|[f]^{p/2}\| \le 1\} \\ &= \left((1 - |\lambda|^2)^{-1/2}\right)^{2/p}. \end{aligned}$$

□

Exercise 5.47 (Neuwirth and Newman, 1967)**.** Let $f \in H^p(\mathbb{D}), p > 0$. Show that $f = \textit{constant}$ if and only if the following hypothesis is verified:

(i) $p \ge 1$ and $f(\zeta)$ is real a.e. $\zeta \in \mathbb{T}$.

(ii) $p \ge 1/2$ and $f(\zeta) \ge 0$ a.e. $\zeta \in \mathbb{T}$.

Show that the conclusion no longer holds if $p < 1$.

Proof. Case (i) is evident, because in this case $f, \overline{f} \in H^1(\mathbb{T})$, which implies f=constant.

For (ii) see Exercise 5.42.

For the last assertion, consider the function $f_1 = i\frac{1+z}{1-z}$ respectively $f_2 = f_1^2$. It follows that $f_1 \in H^p(\mathbb{D})$ for any $p < 1$ and $f_2 \in H^p(\mathbb{D})$ for any $p < 1/2$. □

Exercise 5.48. Let $f, g \in H^2$ and $h = fg$. Show that $|\widehat{h}(n)| \leq \sum_{k+j=n} |\widehat{f}(k)| \cdot |\widehat{g}(j)|$.

Proof. The Fourier series $g = \sum_{j\in\mathbb{Z}} \widehat{g}(z) z^j$ converges in $L^2(\mathbb{T})$ hence by Cauchy Schwartz's inequality the series $h = fg = \sum_{j\in\mathbb{Z}} \widehat{g}(z) f z^j$ converges in $L^1(\mathbb{T})$ and by continuity of $h \mapsto \widehat{h}(n)$, we obtain $\widehat{h}(n) = \sum_{j\in\mathbb{Z}} \widehat{f}(n-j)\widehat{g}(j)$; the result follows. □

Exercise 5.49. Let $\varphi(e^{it}) = i(t-\pi)$ for $0 < t < 2\pi$. Find the Fourier coefficients of φ.

Proof. $\widehat{\varphi}(0) = 0$ and for $k \neq 0$,

$$\begin{aligned}\widehat{\varphi}(k) &= \int_0^{2\pi} i(t-\pi)e^{-ikt} dt/2\pi \\ &= \Big[-(t-\pi)e^{-ikt}/2\pi k\Big]_{t=0}^{2\pi} + \int_0^{2\pi} e^{-ikt} dt/2\pi k \\ &= -1/k\end{aligned}$$

□

Exercise 5.50. (The Hilbert Inequality, 1908) Let $f, g \in H^2$. Show that

$$\left|\sum_{k,j\geq 0} \frac{\widehat{f}(k)\widehat{g}(k)}{k+j+1}\right| \leq \pi \|f\|_2 \|g\|_2.$$

Proof. For $F, G \in L^2(\mathbb{T})$ and $\Phi \in L^\infty(\mathbb{T})$ just as in (a) above, we have $(\Phi F, \overline{G}) = \sum_{i+j+k=0} \widehat{\varphi}(i)\widehat{F}(k)\widehat{G}(j)$, which gives

$$(\varphi f, \overline{zg}) = -\sum_{k,j\geq 0} \frac{\widehat{f}(k)\widehat{g}(j)}{k+j+1}.$$

Then the result follows from

$$|(\varphi f, \overline{zg})| \leq \|\varphi f\|_2 \|\overline{zg}\|_2 \leq \|\varphi\|_\infty \|f\|_2 \|g\|_2 = \pi \|f\|_2 \|g\|_2.$$

□

Exercise 5.51 (The Hardy Inequality, 1926)**.** : For every function $h \in H^1$, $\sum_{k\geq 0} \frac{|\widehat{h}(k)|}{k+1} \leq \pi \|h\|_1$.

Proof. By Exercise 5.45, $h = fg$ with $f, g \in H^2$ and $\|f\|_2^2 = \|g\|_2^2 = \|h\|_1$ and by Exercises 5.48 and 5.50

$$\sum_{k\geq 0} \frac{|\widehat{h}(k)|}{k+1} \leq \sum_{k\geq 0} \frac{\sum_{i+j=k} |\widehat{f}(i)||\widehat{g}(j)|}{k+1} \leq \pi \|f\|_2 \|g\|_2 = \pi \|h\|_1.$$

□

We have seen that every H^p function $f(re^{i\theta})$ converges almost everywhere to an L^p boundary function $f(e^{i\theta})$. It is important to know that whether $f(re^{i\theta})$ always tends to $f(e^{i\theta})$ in the sense of the L^p mean or not.

Exercise 5.52. (Mean convergence theorem) If $f \in H^p(0 < p < \infty)$ then

$$\lim_{r\to 1}\int_0^{2\pi} |f(re^{i\theta})|^p = \int_0^{2\pi} |f(e^{i\theta})|^p \tag{5.9.1}$$

and

$$\lim_{r\to 1}\int_0^{2\pi} \left|f(re^{i\theta}) - f(e^{i\theta})\right|^p d\theta = 0 \tag{5.9.2}$$

Proof. First let us prove 5.9.2 for $p = 2$. If $f(z) = \sum a_n z^n$ is in H^2, then $\sum |a_n|^2 < \infty$. But by Fatou's Lemma

$$\begin{aligned}\int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^2 d\theta \leq& \liminf_{\rho\to 1}\int_0^{2\pi} |f(re^{i\theta}) - f(\rho e^{i\theta})|^2 d\theta\\ =&2\pi\sum_{n=1}^{\infty} |a_n|^2(1-r^n)^2,\end{aligned}$$

which tends to 0 as $r \to 1$. This proves (5.9.2) and hence (5.9.1) for $p = 2$.

If $f \in H^p$ with $0 < p < \infty$, we use the factorization $f = Bg$. Since $[g(z)]^{p/2} \in H^2$, it follows from what we have just proved that

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^p d\theta \to \int_0^{2\pi} |g(e^{i\theta})|^p d\theta = \int_0^{2\pi} |f(e^{i\theta})|^p d\theta$$

This together with the Fatou's Lemma proves (5.9.1)

The following lemma can now be applied to deduce (5.9.2) from (5.9.1). □

Lemma 5.53. *[13][p. 21] Let Ω be a measurable subset of $\mathbb{R}$ and let $\varphi_n \in L^p(\omega), 0 < p < \infty; n = 1, 2, \dots$ As $n \to \infty$, suppose $\varphi_n(x) \to \varphi(x)$ a.e. on Ω and*

$$\int_\Omega |\varphi_n(x)|^p dx \to \int_\Omega |\varphi(x)|^p dx < \infty$$

then

$$\int_\Omega |\varphi_n(x) - \varphi(x)|^p dx \to 0.$$

Corollary 5.54. *If $f \in H^p$ for some $p > 0$, then*

$$\lim_{r\to 1}\int_0^{2\pi} \left|\log^+ |f(re^{i\theta})| - \log^+ |f(e^{i\theta})|\right| d\theta = 0$$

Proof. Immediately follows from Mean convergence theorem 5.52 and the following inequality:

$$|\log^+ a - \log^+ b| \leq \frac{1}{p}|a-b|^p, a \geq 0, b \geq 0, 0 < p \leq 1$$

For the proof the inequality see [13][p. 22] □

Exercise 5.55. [13][p. 34] A function f analytic in $\mathbb{D}$ is representable in the form $f(z) = \mathscr{P}\varphi(z)$ i.e.

$$f(z) := \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) f(e^{it}) dt$$

as a Poisson-integral $\varphi \in L^1$ if and only if $f \in H^1$. In this case $\varphi(t) = f(e^{it})$ a.e.

Proof. If an analytic funcition $f(z)$ has the form $f(z) = \mathscr{P}\varphi(z)$ then

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq \int_0^{2\pi} |\varphi(t)| dt$$

so that $f \in H^1$.

Conversely, suppose $f \in H^1$, and write

$$\Phi(z) := \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) f(e^{it}) dt$$

For any fixed $\rho, 0 < \rho < 1$

$$f(\rho z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) f(\rho e^{it}) dt$$

But by the Exercise 5.52 $\int_0^{2\pi} |f(\rho e^{it}) - f(e^{it})| dt \to 0$ as $\rho \to 1$, so $f(\rho z) \to \Phi(z)$. Hence $\Phi(z) = f(z)$. □

Corollary 5.56. *A function $f(z)$ is analytic in $|z| < 1$ is the Poisson integral of a function $\varphi \in L^p (1 \leq p \leq \infty)$ if and only if $f \in H^p$.*

Chapter 6

Szegö infimum and generalized Phragmén–Lindelöf principle

Using canonical factorization as a structural tool, we analyze Szegő-type extremal and approximation problems. We express the Szegő infimum $\operatorname{dist}(1, H_0^2(\mu))$ *in terms of the logarithmic integral of the weight, describe cyclic vectors and the closure of polynomials, and study the borderline cases. We then introduce the Nevanlinna and Smirnov classes, formulate a conformally invariant framework, and prove a generalized Phragmén–Lindelöf principle adapted to these classes.*

Learning objectives.

- Derive the Szegő formula from factorization and understand its approximation-theoretic meaning.
- Distinguish clearly between H^p, the Nevanlinna class $\mathcal{N}$, and the Smirnov class $\mathcal{N}^+$.
- Use conformal invariance and outer-factor ideas to formulate growth principles on general simply connected domains.

Key ideas.

- Extremal problems in Hardy spaces are governed by logarithmic integrability.
- The Smirnov class is large enough to retain factorization, but still rigid enough to satisfy maximum-principle type statements unavailable in the full Nevanlinna class.

- Conformal transport is not cosmetic: it reveals which arguments are genuinely geometric and which depend on a particular model domain.

Example 6.1 (A Smirnov function outside H^1)**.** The function

$$f(z) = \frac{1}{1-z}$$

belongs to the Smirnov class $N_+(\mathbb{D})$ because its denominator is outer and zero-free on $\mathbb{D}$. However $f \notin H^1(\mathbb{D})$, since its boundary values are not integrable near 1. This example sharply separates N_+ from the Hardy classes.

In this chapter we consider two main applications of the canonical Riesz–Smirnov factorization. First, we express the Szegő infimum $\operatorname{dist}(1, H_0^2(\mu))$ in terms of the measure μ and describe the cyclic vectors of $L^2(\mathbb{T})$. This leads to the classical logarithmic-integral criterion for the completeness of the polynomials, together with the complementary description of the closure $H^2(\mu)$ in terms of the outer function associated with the Radon–Nikodym derivative $w = \frac{d\mu}{dm}$. We also study outer functions, their extremal and extension properties, and boundary-distribution phenomena, before turning to the Smirnov subclass of the Nevanlinna class. After transferring the discussion to arbitrary simply connected domains in $\mathbb{C}$, we obtain a broad Phragmén–Lindelöf-type principle in the spirit of Smirnov (1920) and Helson (1960).

6.1 Szegö infimum and weighted polynomial approximation

Theorem 6.2. (*Szegö, Kolmogorov*) *Let $d\mu = wdm + d\mu_s$ be a Borel measure. Then*

$$\inf_{p \in \mathbb{P}_+^0} \int_{\mathbb{T}} |1-p|^2 d\mu = \exp\left(\int_{\mathbb{T}} \log w \, dm\right).$$

Proof. By Theorem 4.16 two cases are possible

(i) If there exists $f \in H^2$ such that $|f|^2 = w$ a.e. m then $\operatorname{dist}^2 = 0$; otherwise

(ii) $\operatorname{dist}^2 = |\widehat{f}(0)|^2$.

By the Corollary 5.37, Case (ii) $\Leftrightarrow \log w \in L^1$ holds if and only if $\log w \in L^1$ and in this case:

$$f(z) = \exp \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log w^{\frac{1}{2}}(\xi) dm(\xi)$$

Since $f \in H^2$, $\widehat{f}(0) = f(0)$ and $|\widehat{f}(0)|^2 = |f(0)|^2 = \exp \int_{\mathbb{T}} \log w dm$. □

Let $f \in L^2(\mathbb{T})$, and write

$$E_f = \overline{\text{span}}\{z^n f : n \geq 0\}.$$

If $E_f = L^2(\mathbb{T})$, then f is called a *cyclic vector*. Although the positive half of the trigonometric system $(z^n)_{n\geq 0}$ is far from complete in $L^2(\mathbb{T})$ on its own, multiplication by a suitable function f may restore completeness:

$$\overline{\text{span}}\{z^n f : n \geq 0\} = L^2(\mathbb{T}).$$

Corollary 6.3. *Let $f \in L^2(\mathbb{T})$. Then*

$$E_f = \overline{\text{span}}\{z^n f : n \geq 0\} = L^2(\mathbb{T})$$

if and only if $f(\xi) \neq 0$ a.e. on $\mathbb{T}$ and

$$\int_{\mathbb{T}} \log |f| \, dm = -\infty.$$

Proof. Assume first that $E_f = L^2(\mathbb{T})$. Then certainly $zE_f = E_f$. By Wiener's theorem (Theorem 3.5), there exists a measurable set $\sigma \subset \mathbb{T}$ such that

$$E_f = \chi_\sigma L^2(\mathbb{T}).$$

Since $f \in E_f$ and $f \neq 0$ a.e. on $\mathbb{T}$, we must have $\sigma = \mathbb{T}$ up to a null set.

Conversely, assume that $f \neq 0$ a.e. on $\mathbb{T}$ and

$$\int_{\mathbb{T}} \log |f| \, dm = -\infty.$$

If $E_f \neq L^2(\mathbb{T})$, then $zE_f \subsetneq E_f$, because the alternative $zE_f = E_f$ would imply, again by Wiener's theorem, that $E_f = \chi_\sigma L^2(\mathbb{T})$ for some measurable σ, and the condition $f \neq 0$ a.e. would force $\sigma = \mathbb{T}$. Hence $zE_f \subsetneq E_f$, and by the Beurling–Helson theorem there exists a unimodular function θ such that $E_f = \theta H^2$. Since $f \in E_f$, we may write $f = \theta g$ for some $g \in H^2$, so $|f| = |g|$ a.e. on $\mathbb{T}$. Corollary 5.37 then gives $\log |f| \in L^1(\mathbb{T})$, contradicting the assumption that $\int_{\mathbb{T}} \log |f| \, dm = -\infty$. Therefore $E_f = L^2(\mathbb{T})$. □

Example 6.4. (a) If $f(e^{i\theta}) = |1 - e^{i\theta}|^\alpha$, $\alpha > -\frac{1}{2}$, then $E_f \neq L^2(\mathbb{T})$.

(b) If $f(e^{i\theta}) = \exp\left(\frac{-1}{1-e^{i\theta}}\right)$, then $E_f = L^2(\mathbb{T})$.

The following two theorems are final statements on weighted polynomial approximation on the circle $\mathbb{T}$.

Theorem 6.5. *Let μ be a positive measure on $\mathbb{T}$ and let $w = \frac{d\mu}{dm}$ its Radon-Nikodym derivative. Then polynomials $\mathbb{P}_+$ are dense in $L^2(\mu)$ if and only if $\log w \notin L^1(\mathbb{T})$.*

Proof. By Corollary 4.5, the polynomials are dense in $L^2(\mu)$ if and only if the Szegő distance is zero. By Theorem 4.16, this is equivalent to the nonexistence of an outer function f such that $|f|^2 = w$. Corollary 5.37 shows that this happens precisely when $\log w \notin L^1(\mathbb{T})$. □

Theorem 6.6. *Let μ be a positive measure on $\mathbb{T}$, let $d\mu = wdm + d\mu_s$ be its Lebesgue decomposition and suppose that $\log w \in L^1(\mathbb{T})$. Let $\phi \in H^2$ be the outer function defined by $\phi = [w^{\frac{1}{2}}]$. Then closure $H^2(\mu) = \operatorname{clos}_{L^2(\mu)} \mathbb{P}_+$ is given by*

$$H^2(\mu) = L^2(\mu_s) \oplus (\phi^{-1}H^2) = L^2(\mu_s) \oplus \{f \in \mathrm{Hol}(\mathbb{D}) : \ f\phi \in H^2\}.$$

Proof. Indeed, Corollary 4.2 gives $H^2(\mu) = H^2(wdm) \oplus L^2(\mu_s)$ and Lemma 4.4 and Theorem 6.2 show that $H^2(wdm)$ is 1-invariant (non-reducing) subspace of $L^2(wdm)$ (see also Remark 4.3). $\Leftrightarrow$ $H^2(wdm) = \theta H^2$ for some θ such that $|\theta|^2 w = 1$ by the Helson Theorem 3.12 $\implies$ $\theta = [w^{\frac{1}{2}}]^{-1} = \frac{1}{w^{\frac{1}{2}}}$. Hence $H^2(wdm) = \frac{1}{w^{1/2}} H^2 = \varphi^{-1} H^2$ since $\varphi = [w^{1/2}]$. □

6.2 Properties of outer functions

From Theorem 5.28 we see that the condition $\log|f| \in L^1$ is already sufficient to define $[f]$, while the additional assumption $f \in L^p$ guarantees that $[f] \in H^p(\mathbb{D})$. More generally, we make the following definition.

Definition 6.7 (Outer functions)**.** Let h be a measurable function on $\mathbb{T}$ such that $\log|h| \in L^1(\mathbb{T})$. An *outer function* with modulus $|h|$ is a function of the form $f = \lambda[h]$, where $|\lambda| = 1$ and

$$[h](z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log|h(\zeta)|\, dm(\zeta)\right), \qquad |z| < 1.$$

We now record several basic properties of outer functions.

Properties 6.8. (i) An outer function f admits non-tangential boundary limits $\tilde{f}$. Moreover, $f \in H^p(\mathbb{D}) \Leftrightarrow \tilde{f} \in L^p(\mathbb{T})$

Proof. By Fatou's theorem $\lim_{r\to 1} \log |[f]|(r\xi) = \lim_{r\to 1} \int_{\mathbb{T}} P_{r\xi}(\zeta) \log |f|(\zeta) dm(\zeta) = \log |\tilde{f}|(\xi)$ exists non-tangentially a.e. on $\mathbb{T}$. Hence $|[f](\xi)| = |\tilde{f}(\xi)| \implies |[f]| = |\tilde{f}|$.

If $\tilde{f} \in L^p(\mathbb{T})$ then $[f] \in H^p(\mathbb{D})$ follows from the Theorem 5.28 (i). If $[f] \in H^p(\mathbb{D})$ then $\tilde{f} \in L^p$ since $|[f]| = |\tilde{f}|$. □

(ii) Let $f \in H^p, p \geq 1$. Then f is outer if and only if $E_f = \operatorname{clos}_{H^p}(fP_a) = H^p (\Leftrightarrow f$ is cyclic in $H^p)$

(iii) If $f \in H^p$ and $\frac{1}{f} \in H^q$ $(p > 0, q > 0)$, then f is outer.

Proof. $f = \lambda_1 B_1 S_1 [f]$ and $\frac{1}{f} = \lambda_2 B_2 S_2 [\frac{1}{f}] \implies \frac{1}{\lambda_1 B_1 S_1 [f]} = \lambda_2 B_2 S_2 [\frac{1}{f}] \implies 1 = \lambda BS[\frac{f}{f}] = \lambda BS \implies B = 1, S = 1$(since $|B| < 1, |S| < 1$ on $\mathbb{T}$) Similarly, $B_1 = B_2 = 1$ and $S_1 = S_2 = 1$. Hence f is an outer ($\frac{1}{f}$ is also an outer.) □

(iv)

Theorem 6.9 (Smirnov, 1928)**.** *(a) If $f \in \operatorname{Hol}(\mathbb{D})$ and $\operatorname{Re} f(z) \geq 0$ for all $z \in \mathbb{D}$, then $f \in H^p$, $0 < p < 1$ (but perhaps $f \notin H^1(\mathbb{D})$) . Moreover, f is an outer.*

Proof. Note that $\operatorname{Re} f(z) \geq 0, \forall z \in \mathbb{D} \implies \operatorname{Re} f(z) > 0, \forall z \in \mathbb{D}$. Indeed if there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Re} f(z_0) = 0$ then by maximal/minimum principle for harmonic functions $\operatorname{Re} f = 0$ on $\mathbb{D}$, so f is constant, identically equal to 0, a contradiction [see [11] p.150.]

As the values of f are in the right-half plane:

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$$

the function $z \longmapsto (f(z))^p$ is analytic and we can choose $\arg f(z)$ such that $|\arg f(z)^p| \leq p\pi/2$, $z \in \mathbb{D}$. Hence if $0 < p < 1$, then there exists $c_p > 0$ such that $|f(z)|^p \leq c_p \operatorname{Re} f(z)^p$ [since Re $f(z)^p = |f(z)|^p \cos(\arg(f(z))^p)$]. The MVT applied to the harmonic function $\operatorname{Re} f(z)^p$ gives

$$\int_0^{2\pi} |f(re^{i\theta})|^p \frac{dt}{2\pi} \leq \int_0^{2\pi} \operatorname{Re}(f(re^{i\theta})^p)/\cos(\pi p/2) \frac{dt}{2\pi} = \operatorname{Re}(f(0)^p)/\cos(\pi p/2)$$

for $0 \le r < 1$. Hence $f \in H^p(\mathbb{D})$, $0 < p < 1$.

Moreover, since Re $(\frac{1}{f(z)}) \ge 0$ in $\mathbb{D}$, we have f and $\frac{1}{f}$ in H^p, $0 < p < 1$. By Property (iii), f is an outer function. ☐

(b) *More generally, if* $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) \neq 0$ *and* $\alpha := \sum_{z\in\mathbb{D}} |\arg(f(z))| < \infty$ *then* f *is outer and* $f \in H^p(\mathbb{D})$ *for every* $0 < p < \pi/2\alpha$ *(but perhaps* $f \notin H^{\frac{\pi}{2\alpha}}(\mathbb{D})$*.)*

Proof. Apply the first case to $g = f^{\pi/2\alpha}$. ☐

(c) *For every* $h \in L^1(\mathbb{T})$, $\Gamma h \in \cap_{0<p<1} H^p(\mathbb{D})$ *for every* $0 < p < \pi/2\alpha$ *where*

$$\Gamma h(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} h(\zeta) dm(\zeta)$$

Proof. If $h \ge 0$ then Re $\Gamma h(z) \ge 0$ in $\mathbb{D}$, hence $\Gamma h \in \cap_{0<p<1} H^p(\mathbb{D})$. The general case follows from $h = h_1 - h_2 + ih_3 - ih_4$ where $0 \le h_j \le |h|$. ☐

Remark 6.10. By Herglotz's theorem (Theorem 5.29), every function $f \in \mathrm{Hol}(\mathbb{D})$ with $\mathrm{Re}\, f \ge 0$ has the form

$$\Gamma\mu(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) + ic,$$

where μ is a positive measure on $\mathbb{T}$ and $c \in \mathbb{R}$.

Example 6.11. (Herglotz Integral) Let $\mu \in \mathcal{M}(\mathbb{T})$ such that

$$f_\mu = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi).$$

Then $f_\mu \in H^p, 0 < p < 1$ since $\mathrm{Re}\, f_\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2} d\mu = \int_{\mathbb{T}} P_z(\xi) d\mu \ge 0$ if $\mu \ge 0$ and $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ where $0 \le \mu_j \le |\mu|$.

If $\mu \ge 0$ then also $\mathrm{Re}(\frac{1}{f_\mu}) \ge 0 \implies \frac{1}{f_\mu} \in H^p$, hence f_μ is an outer.

Example 6.12. (Cauchy Integral) If f is integrable then $F(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{it} f(e^{it})}{e^{it} - z} dt \implies F(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{it}}{e^{it} - z} d\mu(t)$. If $\mu \ge 0$ then $\mathrm{Re}\{\frac{e^{it}}{e^{it} - z}\} = \frac{1 - r\cos(t-\theta)}{1 - 2r\cos(t-\theta) + r^2} > 0$. Hence $f \in H^p, 0 < p < 1$.

(v) If $f \in H^\infty$ and $\|f\|_\infty \le 1$, then $1 + f$ is outer.

Proof. $\mathrm{Re}(1 + f) \ge 0$ and apply Theorem 6.9 (a) ☐

(vi) The set of outer functions is a commutative group for standard point-wise-point multiplication.

(vii) Let $f,\, g \in H^p (p > 0)$

(a) Then fg is outer if and only if f, g are outer.

Proof. Let $f = \lambda_1 B_1 S_1[f]$ and $g = \lambda_2 B_2 S_2[g]$, hence $fg = (\lambda_1\lambda_2) B_1 B_2 S_1 S_2[fg]$, then use the uniqueness part of the Smirnov Canonical Factorization Theorem 5.33. □

(b) Let f be an outer function and let $|f| \le |g|$, then g is an outer.

Proof. Since $|f| \le |g|$ on $\mathbb{T}$, the quotient f/g belongs to H^∞ and has no zeros in $\mathbb{D}$. By Theorem 5.33, we may write

$$\frac{f}{g} = \lambda S F,$$

where F is outer and S is singular inner. Suppose, for contradiction, that g is not outer. Then $g = \lambda_1 S_1 F_1$, where S_1 is a nontrivial singular inner function and F_1 is outer. Consequently,

$$f = (\lambda\lambda_1)(SS_1)(FF_1),$$

so the factorization of f contains the nonconstant singular inner factor SS_1, contradicting the fact that f is outer. Hence g is outer. □

(c) If $f \in H^p(\mathbb{D})$, $p \ge 1$ and $\inf_{z\in\mathbb{D}} |f(z)| > 0$, then f is outer.

Proof. It follows that for $g \in H^q$ $(q \ge 1)$ we have $\frac{g}{f} \in H^q$ and hence by Theorem 5.34 (iv) f is outer. □

Theorem 6.13. *Let $p > 0$.*

(i) Let $f_n \in H^p$ be a sequence of outer functions with $f_n(0) > 0$. If $|f_n| \searrow$ on $\mathbb{T}$, then $f(z) = \lim_{n\to\infty} f_n(z)$, $z \in \mathbb{D}$ exists uniformly on compact sets. Moreover, if $\lim_{n\to\infty} f_n(0) = 0$, then $f \equiv 0$, otherwise f is an outer H^p function.

(ii) Let $f \in H^p$ be an outer function. Then there exists a sequence of outer functions $f_n \in H^p$ and $\inf_{z\in\mathbb{D}} |f_n(z)| > 0$, $n \ge 1$, $|f_n| \searrow |f|$ on $\mathbb{T}$ (and hence on $\mathbb{D}$) and $f(z) = \lim_{n\to\infty} f_n(z)$, $z \in \mathbb{D}$.

Proof. (i) As the functions f_n are outer, we have

$$\log|f_n(z)| = \int_{\mathbb{T}} P(z\bar{\xi}) \log|f_n(\xi)| dm(\xi).$$

To show the uniform convergence of f_n, it is enough to show that f_n is uniformly Cauchy sequence. For this, we will show $\log|f_n(z)|$ is a uniformly Cauchy.

$$\begin{aligned}\Big|\log|f_n(z)| - \log|f_{n+p}(z)|\Big| &= \left|\int_{\mathbb{T}} P(z\bar{\xi})\, \log\frac{|f_n(\xi)|}{|f_{n+p}(\xi)|}\, dm(\xi)\right| \\ &\le \sup_{|z|\le R} |P(z\bar{\xi})| \int_{\mathbb{T}} \left|\log\frac{|f_n(\xi)|}{|f_{n+p}(\xi)|}\right| dm(\xi) \\ &= C_R\left(\int_{\mathbb{T}} \log|f_n(\xi)|\, dm(\xi) - \int_{\mathbb{T}} \log|f_{n+p}(\xi)|\, dm(\xi)\right).\end{aligned}$$

The conclusion is followed by monotone convergence theorem.

Suppose that $\inf_{n\ge 1} f_n(0) = 0$, then

$$\lim_{n\to\infty} \int_{\mathbb{T}} \log|f_n| dm = \lim_{n\to\infty} \log f_n = -\infty.$$

For a point $z_0 \in \mathbb{D}$, we have $P(z_0\bar{\xi}) \le \frac{1+|z_0|}{1-|z_0|} = C_0$. Hence,

$$\log|f_n(z_0)| \le C_0 \int_{\mathbb{T}} \log|f_n| dm.$$

We conclude that $\lim_{n\to\infty} \log|f_n(z_0)| = -\infty$ and similarly for all $z \in \mathbb{D}$ and we get $f \equiv 0$.

If $\inf_{n\ge 1} f_n(0) > 0$ and $|f_n| \searrow h$ on $\mathbb{T}$, then

$$\int_{\mathbb{T}} \log h dm = \lim_{n\to\infty} \int_{\mathbb{T}} \log|f_n| dm > -\infty,$$

and hence $\log h \in L^1$. Next, it is obvious that $\lim_{n\to\infty} f_n(z) = f(z)$ with $f = [h]$.

(ii) Without loss of generality, we may assume that $f(0) > 0$. Set $f_n = [|f| + \delta_n]$, where $\delta_n > 0$ an appropriate sequence with $\lim_{n\to\infty} \delta_n = 0$ and $\int_{\mathbb{T}} \log(|f| + \delta_n) dm < \infty$. Then f_n satisfies the desired properties.

□

6.3 The Nevanlinna (N) and Smirnov (N_+) classes

Section roadmap.

- Introduce the classes through harmonic majorants and quotient representations.
- Compare them with the Hardy classes and isolate the extra boundary behaviour allowed in N and N_+.
- Prepare the class of functions on which the generalized Phragmén–Lindelöf principle will act.

We know that Nevanlinna class can be represented as

$$N = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \text{ there exist } f_1, f_2 \in \bigcup_{p>0} H^p \text{ such that } f = f_1/f_2 \Big\}$$

and let

$$\mathcal{D} = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \text{ there exist } f_1, f_2 \in \bigcup_{p>0} H^p \text{ such that } f = f_1/f_2 \text{ and } f_2 \text{ is outer} \Big\}$$

be the **Smirnov class** (sometimes denoted by N_+).

Lemma 6.14. *We have*

$$N = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \textit{ there exist } f_1, f_2 \in H^\infty \textit{ such that } f = f_1/f_2 \Big\} \textit{ and}$$

$$\mathcal{D} = \Big\{ f \in \mathrm{Hol}(\mathbb{D}) : \textit{ there exist } f_1, f_2 \in H^\infty \textit{ such that } f = f_1/f_2 \textit{ and } f_2 \textit{ is outer} \Big\}.$$

Proof. Let $f \in N$, $f \not\equiv 0$, and write $f = f_1/f_2$ with $f_1, f_2 \in H^p$ for some $p > 0$. Applying canonical factorization to f itself, write

$$f = \lambda[f] B_1 S_1.$$

Set

$$F_1 := \lambda[\min(1, |f|)] B_1 S_1, \qquad F_2 := [\min(|f|^{-1}, 1)].$$

Then $F_1, F_2 \in H^\infty$, and since $|f| \min(|f|^{-1}, 1) = \min(1, |f|)$, we obtain

$$[|f|]\,[\min(|f|^{-1}, 1)] = [\min(1, |f|)]$$

and hence

$$[f] = [|f|] = \frac{[\min(1, |f|)]}{[\min(|f|^{-1}, 1)]}.$$

Therefore

$$\frac{F_1}{F_2} = \lambda[f]B_1S_1 = f.$$

□

Definition 6.15 (Outer in Nevanlinna class)**.** A function $f \in N$ is called outer if there exist two outer functions f_1, f_2 such that $f = \frac{f_1}{f_2}$.

Properties 6.16. (of the class $\mathcal{D}$ and Nevanlinna outer functions)

(a) If f is outer, then $f \in \mathcal{D}$.

(b) If f_1 and f_2 is outer, then so is $f_1 f_2$.

(c) If $f_1 f_2$ are outer, and $f_1, f_2 \in \mathcal{D}$, then f_1, f_2 are outer.

(d) If $f_1, f_2 \in \mathcal{D}$, then $f_1 f_2 \in \mathcal{D}$.

(e) If $F \in \mathrm{Hol}(\mathbb{D})$, $G \in \mathcal{D}$ and $|F| \leq |G|$ in $\mathbb{D}$, then $F \in \mathcal{D}$.

To verify (c), just let $G = \frac{G_1}{G_2}$ with $G_1, G_2 \in H^\infty$, and G_2 outer. By hypothesis $|G_2 F| \leq |G_1|$ in $\mathbb{D}$, and hence $G_2 F \in H^\infty$. We conclude that $F = \frac{G_2 F}{G_2} \in \mathcal{D}$

Theorem 6.17. (*Generalized Maximum Principle*) *Let $f \in \mathcal{D}$ and g be an outer function in N. If $|f| \leq |g|$ on $\mathbb{T}$, then $|f| \leq |g|$ on $\mathbb{D}$.*

Proof. Let $f = \frac{f_1}{f_2}$ and $g = \frac{g_1}{g_2}$ where f_2, g_1 and g_2 are outer functions in H^∞ and $f_1 \in H^\infty$. By assumption $|f_1 g_2| \leq |f_2 g_1|$ on $\mathbb{T}$ and hence $|f_1 g_2| \leq |[f_1 g_2]| \leq |[f_2 g_1]| = |f_2 g_1|$ in $\mathbb{D}$. □

Remark 6.18. This result is not true in general if $f \in N \setminus \mathcal{D}$ and/or if g is not outer.

Let us recall that by Fatou's theorem every $f \in H^\infty$ has a non-tangential limit a.e. on $\mathbb{T}$ and the boundary function satisfies:

$$\int_{\mathbb{T}} \log |f| dm > -\infty,$$

that means the non-tangential limits of f are non-zero a.e. From here we see that:

Proposition 6.19. *Every function in N class has a non-tangential limit a.e. on $\mathbb{T}$.*

Proposition 6.20. $H^p \subset N_+$

Proof. Hint: If $f \in H^p \backslash \{0\}$ then $f = \lambda BS[f]$ where

$$[f](z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log |f(\zeta)| dm(\zeta)\right).$$

Next $\log = \log^+ - \log^-$ and consider f_1, f_2 corresponding to functions $\log^+$ and $\log^-$. □

Inclusions. $H^p \subset N_+ \subset N$.

Theorem 6.21 (Smirnov theorem)**.** *If $f \in N_+$ and its boundary function belongs to L^p, then $f \in H^p$; equivalently,*

$$N_+ \cap L^p = H^p.$$

Proof. The proof depends on the Arithmetic-Geometric Mean Inequality:

$$\exp\left(\int_{\mathbb{T}} \log h d\sigma\right) \leq \int_{\mathbb{T}} h d\sigma,$$

where h is a non-negative function on $\mathbb{T}$ which is integrable.

If $f \in N^+$ then $f = g_1/g_2$ where $g_1, g_2 \in H^\infty$ and g_2 is outer. Since the presence of an inner factor in g_1 will not affect whether or not $f \in H^2$, we can also assume that g_1 is also an outer. Using the definition of an outer function applied to functions g_1 and g_2 we see that

$$\frac{g_1}{g_2}(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \frac{|g_1(\zeta)|}{|g_2(\zeta)|} dm(\zeta).\right)$$

Furthermore, for each $r \in (0, 1)$ and $w \in \mathbb{T}$

$$\left|\frac{g_1}{g_2}(rw)\right|^2 = \exp\left(\int_{\mathbb{T}} P_{rw}(\zeta) \log \frac{|g_1(\zeta)|^2}{|g_2(\zeta)|^2} dm(\zeta\right)$$

Next apply the Arithmetic-Geometric Mean inequality to the function $|g_1/g_2|$ and the measure $P_{rw}dm$:

$$\left|\frac{g_1}{g_2}(rw)\right|^2 \leq \int_{\mathbb{T}} \left|\frac{g_1}{g_2}(\zeta)\right|^2 P_{rw} dm(\zeta). \tag{6.3.1}$$

Integrate both sides:

$$\begin{aligned}
\int_{\mathbb{T}} |f(rw)|^2 dm(w) &= \int_{\mathbb{T}} \frac{g_1}{g_2}(rw)|^2 dm(w) \\
&\leq \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \left| \frac{g_1}{g_2}(\zeta) \right|^2 P_{rw}(\zeta) dm(\zeta) \right) dm(w) \\
&= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |f(\zeta)|^2 P_{rw}(\zeta) dm(\zeta) \right) dm(w) \\
&= \int_{\mathbb{T}} |f(\zeta)|^2 \left(\int_{\mathbb{T}} P_{rw}(\zeta) dm(w) \right) dm(\zeta) \\
&= \int_{\mathbb{T}} |f|^2 dm
\end{aligned}$$

Thus $\sup_{0<r<1} \int_{\mathbb{T}} |f(rw)|^2 dm(w) \leq \int_{\mathbb{T}} |f|^2 dm$, which implies $f \in H^2$.

To prove the second statement of the theorem, observe that if $f \in N^+$ and $f|_{\mathbb{T}} \in L^\infty$ then as before we can assume $f = g_1/g_2$ and g_1, g_2 are bounded outer functions. By (6.3.1) we see that

$$\begin{aligned}
|f(rw)|^2 = \left| \frac{g_1}{g_2}(rw) \right|^2 &\leq \int_{\mathbb{T}} \left| \frac{g_1}{g_2}(\zeta) \right|^2 P_{rw}(\zeta) dm(\zeta) \\
&= \int_{\mathbb{T}} |f(\zeta)|^2 P_{rw}(\zeta) dm(\zeta) \\
&\leq \|f|_{\mathbb{T}}\|_\infty^2 \int_{\mathbb{T}} P_{rw}\, dm(\zeta) \\
&= \|f|_{\mathbb{T}}\|_\infty^2,
\end{aligned}$$

which implies $f \in H^\infty$. □

Remark 6.22. Smirnov's theorem fails for the full Nevanlinna class N, even for functions analytic on $\mathbb{D}$. For example,

$$f(z) = \exp\left(\frac{1+z}{1-z}\right)$$

is the reciprocal of the atomic inner function described in Example 5.43. It belongs to the Nevanlinna class N, is analytic on $\mathbb{D}$, and has boundary values of unit modulus a.e. on $\mathbb{T}$. However, it does not belong to H^2, because, as noted in Remark 5.44,

$$|f(r)| = \exp\left(\frac{1+r}{1-r}\right), \qquad r \in (0,1),$$

which violates the necessary growth estimate for an H^2 function (see [8, p. 59]):

$$|f(\lambda)| \leq \frac{\|f\|}{\sqrt{1-|\lambda|^2}}, \qquad f \in H^2.$$

The original definition of the Nevanlinna class differs from Definition 6.3. One has $f \in N$ if and only if

$$\sup_{0\leq r<1} \int_{\mathbb{T}} \log^+ |f(r\xi)|\, d\xi < \infty.$$

The equivalence of the two definitions is far from obvious; proofs may be found in Nevanlinna and Nevanlinna (1922), Privalov (1941), Duren (1970) [13, p. 16], and Koosis (1998) [4]. We record the theorem in the following form.

Theorem 6.23. *[13, p. 16] A function analytic in the unit disk belongs to the class N if and only if it can be written as the quotient of two bounded analytic functions.*

Proof. ($\Leftarrow$) Suppose first that $f(z) = \varphi(z)/\psi(z)$ where φ, ψ are analytic and bounded in $\mathbb{D}$. There is no loss of generality in assuming $|\varphi(z)| \leq 1, |\psi(z)| \leq 1$ and $\psi(0) \neq 0$. Then

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \leq -\int_0^{2\pi} \log |\psi(re^{i\theta})| d\theta.$$

But by Jensen's formula (see Ahlforse, p. 206)

$$\frac{1}{2\pi}\int_0^{2\pi} \log |\psi(re^{i\theta})| d\theta = \log |\psi(0)| + \sum_{|x_n|<r} \log \frac{r}{|z_n|},$$

where z_n are zeros of ψ. This shows that $\int \log|\psi|$ increases with r, so $f \in N$.

($\Rightarrow$) Let $f(z) \not\equiv 0$ be of class N. Let f has a zero of multiplicity $m \geq 0$ at the origin, so that $z^{-m} f(z) \to \alpha \neq 0$ as $z \to 0$. Let z_n be the other zeros of f, repeated according to multiplicity and arranged so that $0 < |z_1| \leq |z_2| \leq \cdots < 1$. If $f(z) \neq 0$ on the circle $|z| = \rho < 1$, the function

$$F(z) = \log \left\{ f(z) \frac{\rho^m}{z^m} \Pi_{|z_n|<\rho} \left(\frac{\rho^2 - \overline{z}_n z}{\rho(z - z_n)} \right) \right\}$$

is analytic in $|z| \leq \rho$, and $\operatorname{Re} F(z) = \log |f(z)|$ on $|z| = \rho$. Hence by analytic completion of the Poisson formula:

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\rho e^{it})| \frac{\rho e^{it} + z}{\rho e^{it} - z} dt + iC.$$

This is sometimes called the Poisson-Jensen formula. After exponentiation, it takes of the form $f(z) = \varphi_\rho(z)/\psi_\rho(z)$ where

$$\varphi_\rho(z) = \frac{z^m}{\rho^m}\Pi_{|z_n|<\rho}\frac{\rho(z-z_n)}{\rho^2-\overline{z}_n z}.\exp\left\{-\frac{1}{2\pi}\int_0^{2\pi}\log^-|f(\rho e^{it})|\frac{\rho e^{it}+z}{\rho e^{it}-z}dt+iC\right\}$$

$$\psi_\rho(z) = \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(\rho e^{it})|\frac{\rho e^{it}+z}{\rho e^{it}-z}dt\right\}$$

Next choose a sequence $\{\rho_k\}$ increasing to 1 such that $f(z) \neq 0$ on the circles $|z| = \rho_k$. Let $\Phi_k(z) = \varphi_{\rho_k}(\rho_k z;); \Psi_k(z) = \psi_{\rho_k}(\rho_k z)$. Then $f(\rho_k z) = \Phi_k(z)/\Psi_k(z)$ in $\mathbb{D}$. But the functions are analytic in the unit disk, and $|\Phi_k(z)| \leq 1$, $|\Psi_k(z)| \leq 1$. Hence $\{\Phi_k\}$ and $\{\Psi_k\}$ are normal families, and there exists a sequence $\{k_i\}$ such that $\Phi_{k_i}(z) \to \varphi(z)$ and $\Psi_{k_i}(z) \to \psi(z)$ uniformly in each disk $|z| \leq R < 1$. The function φ, ψ are analytic in unit disk and $|\varphi(z)| \leq 1, |\psi(z)| \leq 1$. According to the definition of ψ_ρ the fact that $\int \log^+|f|$ is bounded gives a uniform estimate $|\Psi_k(0)| \geq \delta > 0$, so $\psi(z) \not\equiv 0$. Thus $f = \varphi/\psi$ and the proof is completed. □

This theorem is important because it allows properties of functions in N to be deduced from the corresponding properties of bounded analytic functions. In particular, it gives access to their boundary behavior.

Theorem 6.24. *For each $f \in N$, the non-tangential limit $f(e^{i\theta})$ exists almost everywhere and $\log|f(e^{i\theta})|$ is integrable unless $f(z) \equiv 0$. If $f \in H^p$ for some $p > 0$, then $f(e^{i\theta}) \in L^p$.*

Proof. Assuming $f(z) \not\equiv 0$, represent in the form $\varphi(z)/\psi(z)$, where $|\varphi(z)| \leq 1$ and $|\psi(z)| \leq 1$. Since φ and ψ are bounded analytic functions, they have non-tangential limits $\varphi(e^{i\theta})$ and $\psi(e^{i\theta})$ almost everywhere. Appleaing to Fatou's Lemma we have

$$\int_0^{2\pi}|\log|\varphi(e^{i\theta})||d\theta \leq \liminf_{r\to 1}\left\{-\int_0^{2\pi}\log|\varphi(re^{i\theta})d\theta\right\}$$

But $\int \log\varphi(re^{i\theta})d\theta$ increases with r, by Jensen's theorem. Hence $\log|\varphi(e^{i\theta})| \in L^1$ and similarly for ψ. In particular $\psi(e^{i\theta})$ cannot vanish on a set of positive measure. The radial limit $f(e^{i\theta})$ therefore exists almost everywhere, and $\log|f(e^{i\theta})| \in L^1$. Another application of Fatou's lemma shows that $f(e^{i\theta}) \in L^p$ if $f \in H^p$. □

The theorem says that if $f \in N$ and if $f(e^{i\theta}) = 0$ on a set of positive measure, then $f(z) \equiv 0$. In other words, a function of class N is uniquely determined by its boundary values on any set of positive measure.

It is evident from the representation $f = \varphi/\psi$ that $\int \log^- |f(re^{i\theta})| d\theta$ is bounded if $f \in N$. Hence $f \in N$ if and only if $\int \left|\log |f(re^{i\theta})|\right| d\theta$ is bounded.

6.4 A conformally invariant framework

Here we consider the classes $\mathrm{Nev}(\Omega)$ and $\mathcal{D}(\Omega)$, where Ω is a simply connected domain ($\neq \mathbb{C}$), that is, domains that are conformally equivalent to the open unit $\mathbb{D}$.

Definition 6.25. Define

$$H^\infty(\Omega) = \{f \in \mathrm{Hol}(\Omega) : \|f\|_{H^\infty} = \sup_{z\in\Omega} |f(z)| < \infty\}$$

and

$$N(\Omega) = \{f \in \mathrm{Hol}(\Omega) : \text{ there exist } f_1, f_2 \in H^\infty(\Omega) \text{ such that } f = f_1/f_2\}\,.$$

For $\omega : \mathbb{D} \to \Omega$ be an onto conformal map. A function $f \in \mathrm{Nev}(\Omega)$ is called outer if $f \circ w$ is an outer in $\mathrm{Nev}(\mathbb{D})$. With this definition, we get

$$\mathcal{D}(\Omega) = \Big\{f \in \mathrm{Hol}(\Omega) : \text{ there exist } f_1, f_2 \in H^\infty(\Omega) \text{ such that } f = f_1/f_2 \text{ and } f_2 \text{ is outer}\Big\}.$$

The following two results are straightforward transfers to Ω of well-known factorization statements on $\mathbb{D}$. Note that if $\omega : \Omega \to \mathbb{D}$ extends to a homeomorphism of $\mathrm{clos}(\Omega)$ onto $\mathrm{clos}(\mathbb{D})$, then we call Ω a **Jordan domain**.

Lemma 6.26. (*Generalized Maximum Principle*) *Let Ω be a Jordan domain. Let $\lambda \in \partial\Omega$, $f \in \mathcal{D}(\Omega) \cap C\Big(\mathrm{clos}(\Omega) \setminus \{\lambda\}\Big)$ and let g be an outer function such that $g \in C\Big(\mathrm{clos}(\Omega) \setminus \{\lambda\}\Big)$ and $|f| \le |g|$ on $\partial\Omega \setminus \{\lambda\}$. Then $|f| \le |g|$ on Ω.*

Lemma 6.27. *Let $f \in H^\infty(\Omega)$. Then f is outer if and only if there exists a sequence of outer functions $(f_n)_{n\ge1} \in H^\infty(\Omega)$ such that*

$$\inf_{z\in\Omega} |f_n(z)| > 0,\ n \ge 1,\ \ \lim_{n\to\infty} f_n(z) = f(z),\ \ |f_n(z)| \searrow |f(z)|, z \in \Omega.$$

Corollary 6.28. *Let $\Omega_1 \subset \Omega_2$ be two simply connected domains and let $f \in N(\Omega_2)$.*

(i) If f is outer on Ω_2, then $f|_{\Omega_1}$ is outer on Ω_1.

(ii) If $f \in \mathcal{D}(\Omega_2)$, then $f|_{\Omega_1} \in \mathcal{D}(\Omega_1)$.

6.5 The generalized Phragmén–Lindelöf principle

Theorem 6.17 and Lemma 6.26 may be viewed as versions of the Phragmén–Lindelöf principle. The essential point is that, in general, the relevant majorants need not be analytic functions.

Let Ω be a Jordan domain, let M and M_* be two non-negative functions on Ω, and let $\omega \in C(\partial\Omega \setminus \{\lambda\})$, where $\lambda \in \partial\Omega$. Then M_* is called a Phragmén–Lindelöf majorant for M and ω if, for every

$$f \in \mathrm{Hol}(\Omega) \cap C(\mathrm{clos}(\Omega) \setminus \{\lambda\})$$

with $|f| \le M$ on Ω and $|f| \le \omega$ on $\partial\Omega \setminus \{\lambda\}$, one has $|f| \le M_*$ on Ω.

Theorem 6.29. (*Generalized Phragmén–Lindelöf principle*) *Let $F \in \mathcal{D}(\Omega)$ and $G \in N(\Omega) \cap C(\mathrm{clos}(\Omega) \setminus \{\lambda\})$ be such that $M \le |F|$ on Ω and $\omega \le |G|$ on $\partial\Omega \setminus \{\lambda\}$. Then either there exists an outer function $[\omega \circ \omega]$ (in which case*

$$M_* = |[\omega \circ \omega] \circ \omega^{-1}|$$

is a Phragmén–Lindelöf majorant for M and ω), or every function

$$f \in \mathrm{Hol}(\Omega) \cap C(\mathrm{clos}(\Omega) \setminus \{\lambda\})$$

with $|f| \le M$ on Ω and $|f| \le \omega$ on $\partial\Omega \setminus \{\lambda\}$ is identically zero (and then $M_* = 0$).

Proof. In view of part (e) of Properties 6.16, the inequalities $|F| \le M \le |F|$ show that $M \in \mathcal{D}(\Omega)$. If there exists $f \not\equiv 0$ in $N(\Omega)$ such that

$$|f \circ \omega| \le \omega \circ \omega \le |G \circ \omega|$$

on $\mathbb{T} \setminus \omega^{-1}(\{\lambda\})$, then we can define the outer function $[\omega \circ \omega]$. Applying Lemma 6.26, we obtain

$$|f \circ \omega| \le |[\omega \circ \omega]|$$

on $\mathbb{T} \setminus \omega^{-1}(\{\lambda\})$, and the conclusion follows. □

6.6 Exercises and extensions

The following exercises isolate two useful tests for outerness that complement the general structural theorems proved above.

Further exercises

Exercise 6.30. Let b be a non-constant function in the closed unit ball of H^∞. Put

$$f = \frac{1}{1-b}.$$

Then f is outer in H^p for $0 < p < 1$.

Proof. Since $b \in \{\|f\|_\infty \le 1 : f \in H^\infty\}$ by the maximal principle, $|b(z)| < \|b\|_\infty \le 1$ for each $z \in \mathbb{D}$. Hence f is analytic on $\mathbb{D}$. Moreover we have

$$\operatorname{Re} \frac{1}{1-b(z)} = \frac{1 - \operatorname{Re} b(z)}{|1-b(z)|^2} \ge \frac{1-|b(z)|^2}{|1-b(z)|^2} > 0 \quad (z \in \mathbb{D}).$$

Hence by Smirnov Theorem 6.9, f is an outer in H^p. □

Exercise 6.31. If a polynomial p has no zero in the open disk $\mathbb{D}$, then p is outer.

Proof. Consider $p(z) = \text{const} \prod_{i=1}^{n} \left(1 - \frac{z}{\xi_i}\right)$, $|\xi_i| \ge 1$. As $|z| < 1$ and $|\xi_i| \ge 1$, we have $\operatorname{Re}\left(1 - \frac{z}{\xi_i}\right) \ge 0$. By applying Theorem 6.9 and the Property ((a).) □

Chapter 7

Harmonic analysis in $L^2(\mathbb{T}, \mu)$

This chapter connects Hardy-space structure to classical harmonic analysis on the circle. We study skew projections and bases of exponentials in weighted L^2 spaces, develop the Riesz projection and harmonic conjugation, and record several equivalent formulas for the conjugate function. The main highlight is the Helson–Szegő theorem, which characterizes those weights for which Fourier series converge in $L^2(\mathbb{T}, \mu)$.

Learning objectives.

- Relate harmonic conjugation, the Hilbert transform, and the Riesz projection in a unified framework.
- Understand how weighted L^2 estimates control Fourier convergence and basis properties.
- Interpret the Helson–Szegő theorem as a structural criterion for well-behaved generalized Fourier series.

Key ideas.

- The Hilbert transform is the real-variable shadow of analytic projection.
- Boundedness of the Riesz projection on weighted spaces encodes subtle geometric information about the weight.
- Exponential systems, singular integrals, and Hardy-space factorization are three facets of the same harmonic-analytic structure.

Example 7.1 (The first Hilbert-transform identities)**.** For every integer $n \ge 1$,

$$\widetilde{\cos(nt)} = \sin(nt), \qquad \widetilde{\sin(nt)} = -\cos(nt),$$

up to the normalization adopted for zero-mean conjugates. These identities explain why harmonic conjugation acts diagonally on Fourier modes and why the Riesz projection can be written in terms of the Hilbert transform.

The main result of this section is the Helson–Szegö theorem characterizing those $L^2(\mathbb{T},\mu)$ in which the Fourier series of every function $f \in L^2(\mathbb{T},\mu)$ converges in the norm topology. This is one of the main results of harmonic analysis on the circle group $\mathbb{T}$. It is closely related to generalized Fourier series with respect to a minimal sequence; harmonic conjugates, the Riesz projections, and weighted estimates for Hilbert singular integrals.

Definition 7.2. A sequence $(x_n)_{n\ge1}$ in a Banach space X is called **minimal** if

$$x_n \notin M_n := \overline{\operatorname{span}}\{x_k : k \ne n\}$$

for every $n \ge 1$. It is called **uniformly minimal** if

$$\inf_{n\ge1} \operatorname{dist}\Big(\frac{x_n}{\|x_n\|}, M_n\Big) > 0.$$

To proceed, we need a standard corollary of the Hahn–Banach theorem.

Proposition 7.3. *Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$. Then $x_0 \in \overline{M}$ if and only if there does not exist a bounded linear functional f on X such that $f(x) = 0$ for all $x \in M$ but $f(x_0) \ne 0$ (indeed, one may arrange $f(x_0) = 1$).*

Proof. ($\Longleftarrow$) If $x_0 \in \overline{M}$ and f is a bounded linear functional on X satisfying $f(x) = 0$ for all $x \in M$, then continuity gives $f(x_0) = 0$.

($\Longrightarrow$) Suppose $x_0 \notin \overline{M}$. Then there exists $\delta > 0$ such that $\|x - x_0\| > \delta$ for every $x \in M$. Let M' be the subspace generated by M and x_0, and define $f : M' \to \mathbb{C}$ by

$$f(x + \lambda x_0) = \lambda, \qquad x \in M,\ \lambda \in \mathbb{C}.$$

Then $f(x) = 0$ on M and $f(x_0) = 1$. Moreover,

$$\delta|\lambda| \le \|\lambda x_0 + x\|,$$

and therefore

$$|f(x+\lambda x_0)| = |\lambda| \leq \frac{1}{\delta}\|\lambda x_0 + x\|.$$

Thus f is bounded on M'. By the Hahn–Banach theorem, f extends to a bounded linear functional $\tilde{f}$ on X. □

Lemma 7.4. *(i) A sequence $(x_n)_{n\geq 1} \subset X$ is minimal if and only if there exist functionals $f_n \in X^*$ such that $(x_k, f_n) = \delta_{kn}$. Such a pair $((x_n)_{n\geq 1}, (f_k)_{k\geq 1})$ is called* biorthogonal*, and the functionals f_n are the corresponding coordinate functionals.*

(ii) $(x_n)_{n\geq 1} \subset X$ is uniformly minimal if and only if there exists a sequence $(f_n)_{n\geq 1}$ of coordinate functionals such that $\sup_{n\geq 1} \|x_n\|\,\|f_n\| < \infty$.

Proof. (i) By Hahn-Banach's theorem, if $x_n \notin M_n$, then there exists a sequence $f_n \in X^*$ with $\|f_n\| = 1$, $f_n(x_n) = \|x_n\|$, $\tilde{f}_n(x_n) = 1$, $\tilde{f}_n = \frac{f_n}{\|x_n\|}$.

(ii) Moreover for any subspace $E \subset X$,

$$\operatorname{dist}(x, E) = \sup\{|f(x)| : f \in X^*, \ f|_{\mathbb{E}} \equiv 0, \ \|f\| \leq 1\}.$$

For this, if $x \in E$ then both sides are equal. So firstly we will show "$\leq$". When $x \notin E$, by Hahn- Banach's theorem there exists $\tilde{f} \in X^*$ such that $\tilde{f}(x) = \operatorname{dist}(x, E)$, and $\tilde{f}(E) = 0$ with $\|\tilde{f}\| \leq 1$. Implies

$$\operatorname{dist}(x, E) = |\tilde{f}(x)| \leq \sup\{|f(x)| \ : \ f \in X^*, \ f|_{\mathbb{E}} \equiv 0, \ \|f\| \leq 1\}.$$

For the other inequality, let $y \in E$, then we have

$$|f(x)| = |f(x-y)| \leq \|f\|\|x-y\| \leq \|x-y\|,$$

and hence $|f(x)| \leq \inf_{y\in E} \|x-y\| = \operatorname{dist}(x, E)$. This implies

$$\sup\{|f(x)| : f \in X^*, \ f|_{\mathbb{E}} \equiv 0, \ \|f\| \leq 1\} \leq \operatorname{dist}(x, E).$$

Thus,

$$\sup\{|f(x)| : f \in X^*, \ f|_{\mathbb{E}} \equiv 0, \ \|f\| \leq 1\} = \operatorname{dist}(x, E).$$

Next, replacing f by $f/f(x)$, it follows that

$$\inf\Big\{\|f\| : f \in X^*, \ f|_{\mathbb{E}} \equiv 0, \ f(x) = 1\} = \frac{1}{\operatorname{dist}(x, E)}.$$

(If $\phi \neq S \subset (0, \infty)$ then $\frac{1}{\sup(S)} = \inf \frac{1}{S} = \inf_{s \in S} \frac{1}{s}$)

Main Proof: Apply this to $x = x_n$, $E = M_n$, and let $f_n \in X^*$ be the corresponding coordinate functionals with minimal norm. Then,

$$\operatorname{dist}\left(\frac{x_n}{\|x_n\|}, M_n\right) = \frac{1}{\|x_n\|} \operatorname{dist}(x_n, M_n) = \frac{1}{\|x_n\|} \frac{1}{\|f_n\|}.$$

Thus,

$$\inf_{n \geq 1} \operatorname{dist}\left(\frac{x_n}{\|x_n\|}, M_n\right) > 0 \text{ if and only if} \sup_{n \geq 1} \|x_n\| \, \|f_n\| < \infty.$$

□

Definition 7.5. To a minimal sequence (x_n) we associate the (formal) Fourier series

$$x \sim \sum_{n \geq 1} (x, f_n) x_n, \; x \in X.$$

The operator $x \mapsto P_n x = (x, f_n) x_n$ is called the projection onto the nth Fourier component (or the coordinate projection associated with the biorthogonal pair $((x_n)_{n \geq 1}, (f_k)_{k \geq 1})$).

Remark 7.6. We have $\|P_n\| = \|f_n\| \|x_n\|$ (because $f_n(x_n) = 1$).

Proof. $\|P_n(x_n)\| = |f_n(x_n)| \|x_n\| = 1.\|x_n\| = \|f_n\| \|x_n\|$ (since $f_n(x_n) = 1$, and $1 = \|f_n\|$). Also, since $P_n x = (x, f_n) x_n$ we have

$$\sup_{x \neq 0} \frac{\|P_n(x)\|}{\|x\|} \leq \|f_n\| \|x_n\|$$
$$\implies \|P_n\| = \|f_n\| \|x_n\|,$$

because equality is attained at the point x_n. □

Definition 7.7. A sequence (x_n) in a Banach space X is called a **basis** of X if for every $x \in X$ there exists a unique sequence $(a_n) \subset \mathbb{C}$ such that

$$x = \sum_{k \geq 1} a_k x_k.$$

We write $a_n = a_n(x)$ for the corresponding coefficients. A sequence (x_n) is called a **basis sequence** if it is a basis for its closed linear span $\overline{\operatorname{span}}_X \{x_n : n \geq 1\}$.

Theorem 7.8. (*S. Banach, 1932*) *Let (x_k) be a basis of the Banach space X. Then (x_k) is uniformly minimal and $f_k(x) = a_k(x)$, $x \in X$ are the coordinate functionals.*

Definition 7.9. Let X be a Banach space and let $(x_n)_{n\in\mathbb{Z}}$ be a family in X. It is called a **symmetric basis** if for every $x \in X$ there exists a unique sequence $(a_k(x))_{k\in\mathbb{Z}} \subset \mathbb{C}$ such that

$$x = \lim_{n\to\infty} \sum_{k=-n}^{n} a_k(x) x_k.$$

It is called **non-symmetric** if

$$x = \lim_{m,n\to\infty} \sum_{k=-m}^{n} a_k(x) x_k.$$

Lemma 7.10. *Let $\chi = (x_k)_{k\in\mathbb{Z}}$ and $(f_k)_{k\in\mathbb{Z}}$ be a biorthogonal pair in a Banach space X. Set $P_{m,n} = \sum_{k=-m}^{n} (\,.\,, f_k) x_k,\ \ m, n \in \mathbb{Z}$. Then*

(i) χ is a symmetric (respectively non-symmetric) basis if and only if $\sup_{n\geq 1} \|P_{-n,n}\| < \infty$ (respectively $\sup_{m,n} \|P_{m,n}\| < \infty$) and χ is complete.

(ii) If χ is a (at least symmetric) basis, then $(f_k)_{k\in\mathbb{Z}}$ is total, i.e. $f_k(x) = 0$ for all $k \in \mathbb{Z}$ implies $x = 0$.

(iii) For $\sigma \subset \mathbb{Z}$, define $\chi_\sigma = \overline{\text{span}}\{x_k : k \in \sigma\}$ and $\chi^\sigma = \overline{\text{span}}\{x \in X : f_k(x) = 0 \text{ for all } k \notin \sigma\}$. If χ is a basis, then for all $\sigma \subset \mathbb{Z}$, we have $\chi_\sigma = \chi^\sigma$.

Proof. (i) Since χ is a basis, $\lim_{m,n} P_{m,n} x = x$ for all $x \in \mathcal{L}in\{x_k : k \in \mathbb{Z}\}$. By the UBP (uniform bounded principle: pt-wise bounded implies uniform bounded) $\sup_{m,n} \|P_{m,n}\| < \infty$.

(ii) If $f_k(x) = 0$ for all $k \in \mathbb{Z}$, then $P_{-n,n} x = 0$ for all $n \geq 1$. Hence $x = 0$.

(iii) The inclusion $\chi_\sigma \subset \chi^\sigma$ is clear (even for minimal families). On the other hand, if $x \in X^\sigma$, then $x = \lim_{n\to\infty} P_{-n,n} x$ with $P_{-n,n} x \in X_\sigma$. Hence $x \in X_\sigma$.

□

7.1 Skew projections

Let L, M be two subspaces of a vector space X such that $L \cap M = \{0\}$. Define $P : L + M \to X$ by $P(x + y) = x$, then $P^2 = P$, $P|_L = id$ and $P|_M = 0$. Then P is called **skew projection** onto L parallel to M and denoted as $P := P_{L||M}$.

Lemma 7.11. *Let L, M be two subspaces of a Banach space X verifying $L \cap M = \{0\}$. Then*

(i) $P_{L||M}$ is continuous if and only if $P_{\bar{L}||\bar{M}}$ is well defined and continuous (here $\bar{L} =$ clos L and $\bar{M} =$ clos M).

Proof. Let $x + y \in L + M, x \in L, y \in M$. Then $P_{L||M}$ is continuous $\iff \|P_{L||M}(x + y)\| = \|x\| \leq c\|x+y\|$ for every $x \in L, y \in M \iff \|\overline{x}\| \leq C\|\overline{x}+\overline{y}\|, \overline{x} \in L,$ and $\overline{y} \in M$ $\iff P_{\overline{L}||\overline{M}}$ is continuous. □

(ii) If L, M are closed, then $P_{L||M}$ is continuous if and only if $L + M =$ clos $(L + M)$.

Proof. Apply closed graph theorem for the operator $T = P_{L||M}$. □

Definition 7.12. Let L, M be two subspaces of a Hilbert space H. Define angle $\alpha \in [0, \frac{\pi}{2}]$ (or minimal angle) between L and M by

$$\cos\langle L, M\rangle = \cos\alpha = \sup_{x\in L,\, y\in M} \frac{|\langle x, y\rangle|}{\|x\|\|y\|}.$$

NOTATION: We write $\alpha = \langle L, M\rangle$.

Remark 7.13. $L \perp M$ if and only if $\alpha = \frac{\pi}{2}$.

Lemma 7.14. *With the above notations we have*

$$\cos\langle L, M\rangle = \cos\langle \bar{L}, \bar{M}\rangle = \|P_{\bar{M}}P_{\bar{L}}\|$$

and

$$\sin\langle L, M\rangle = \sin\langle \bar{L}, \bar{M}\rangle = \|P_{L||M}\|^{-1},$$

where the symbols have obvious meaning.

Proof. By definition, $\sup_{y\in M\setminus\{0\}} \frac{|(P_{\bar{M}}x, y)|}{\|y\|} = \|P_{\bar{M}}x\|$. Moreover, $\langle x, y\rangle = \langle P_{\bar{M}}x, y\rangle$ for $y \in M$ and hence

$$\begin{aligned}\cos\langle L, M\rangle = \sup_{0\neq x\in L,\, 0\neq y\in M} \frac{|\langle x, y\rangle|}{\|x\|\|y\|} &= \sup_{0\neq x\in L,\, 0\neq y\in M} \frac{|\langle P_{\overline{M}}x, y\rangle|}{\|x\|\|y\|} \\ &= \sup_{0\neq x\in L} \frac{1}{\|x\|} \sup_{0\neq y\in M} \frac{|\langle P_{\overline{M}}x, y\rangle|}{\|y\|} \\ &= \sup_{0\neq x\in L} \frac{\|P_{\bar{M}}x\|}{\|x\|}.\end{aligned}$$

But
$$\sup_{0\neq x\in L}\frac{\|P_{\bar{M}}x\|}{\|x\|}=\sup_{0\neq x\in L}\frac{\|P_{\bar{M}}P_{\bar{L}}x\|}{\|x\|}=\sup_{0\neq x\in H}\frac{\|P_{\bar{M}}P_{\bar{L}}x\|}{\|x\|}=\|P_{\bar{M}}P_{\bar{L}}\|.$$
Hence $\cos\langle L,M\rangle=\|P_{\bar{M}}P_{\bar{L}}\|$

Next,

$$\begin{aligned}\|P_{L||M}\|^2&=\sup_{0\neq x\in L,\,0\neq y\in M}\frac{\|P_{L||M}(x+y)\|^2}{\|x+y\|^2}\\&=\sup_{0\neq x\in L,\,0\neq y\in M}\frac{\|x\|^2}{\|x+y\|^2}\\&=\sup_{0\neq x\in L}\frac{\|x\|^2}{\inf_{0\neq y\in M}\|x+y\|^2}\\&=\sup_{0\neq x\in L}\frac{\|x\|^2}{\|(1-P_{\bar{M}})x\|^2}.\end{aligned}$$

This now gives

$$\sin^2\langle L,M\rangle=1-\cos^2\langle L,M\rangle=1-\sup_{0\neq x\in L}\frac{\|P_{\bar{M}}x\|^2}{\|x\|^2}=\inf_{0\neq x\in L}\frac{\|(1-P_{\bar{M}})x\|^2}{\|x\|^2}=\frac{1}{\|P_{L||M}\|^2.}$$

So $\sin\langle L,M\rangle=\frac{1}{\|P_{L||M}\|}$. □

Corollary 7.15. *The projection $P_{L||M}$ is continuous if and only if $\|P_{\bar{L}}P_{\bar{M}}\|<1$ (and hence if and only if $\langle L,M\rangle>0$). Moreover, $\|P_{L||M}\|=\|P_{M||L}\|$.*

Proof. $P_{L||M}$ is continuous $\Leftrightarrow$ $\|P_{L||M}\|$ exists and >0 $\Leftrightarrow$ $\frac{1}{P_{L||M}}$ exists and >0 $\Leftrightarrow$ $\sin\langle L,M\rangle>0\Leftrightarrow\langle L,M\rangle>0$. Since $\sin\langle L,M\rangle>0\Leftrightarrow\cos\langle L,M\rangle<1\Leftrightarrow\|P_{\overline{M}}P_{\overline{L}}\|<1$ by Lemma 7.14 □

7.2 Bases of exponentials in $L^2(\mathbb{T},\mu)$

Now, let $X=L^2(\mathbb{T},\mu)$, where μ is a finite Borel measure, and $x_k=e^{ikt}$, $k\in\mathbb{Z}$ (or, $x_k=z^k$, $k\in\mathbb{Z}$).

Lemma 7.16. *If $(e^{ikt})_{k\in\mathbb{Z}}$ is a basis of $L^2(\mu)$ then $\mu_s\equiv 0$.*

Proof. Let $\sigma_n=\{k:\ k>n\}$, let $L^2_{\sigma_n}=\overline{\text{span}}_{L^2(\mu)}\{z^k:\ k>n\}$, and let f_k be coordinate functionals associated to $(e^{ikt})_{k\in\mathbb{Z}}$, then

$$\bigcap_{n\geq 1}L^2_{\sigma_n}=\{x\in L^2(\mu):f_k(x)=0\text{ for all }k\in\mathbb{Z}\}=\{0\}$$

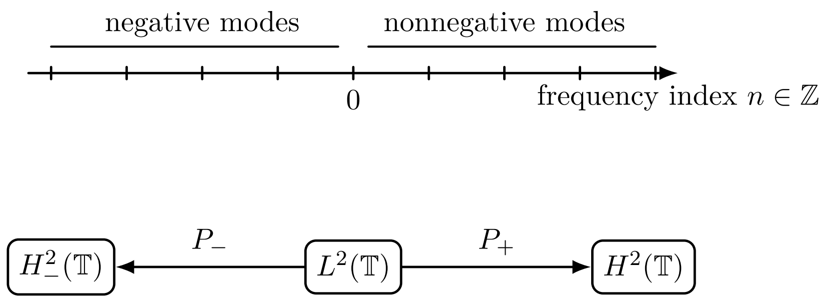

Figure 7.1: Fourier-frequency picture behind the Riesz projections: P_+ keeps the nonnegative Fourier modes and P_- keeps the negative modes.

($\because$ $x \in L^2(\mu) \implies x = \sum_{k\in\mathbb{Z}}\langle x, f_k\rangle z^k = \sum_{k\in\mathbb{Z}} f_k(x) z^k$ and $f_k(x) = 0$ since $f_k \perp L^2(\sigma_k)$ for all $k \geq 1$(by Proposition 7.3) $\implies$ $x = 0$ (by Banach's theorem 7.8). Consequently, $L^2_{\sigma_n}$ is an invariant subspace, and $z^n \in L^2_{\sigma_n}$ and $z^n \neq 0$ on $\mathbb{T}$. Thus it can be deduced (as in Corollary 4.2) that $L^2_{\sigma_n} = L^2_{\sigma_n}(\mu_a) + L^2(\mu_s)$ for all $n \in \mathbb{Z}$. But then also $\bigcap_{n\geq 1} L^2_{\sigma_n} \supset L^2(\mu_s)$, implies $L^2(\mu_s) = 0$. □

Remark 7.17. For studying exponential basis in $L^2(\mathbb{T},\mu)$ one can restrict to measure which is absolutely continuous with respect to the Lebesgue measure m, $d\mu = wdm$, $w \in L^1_+(\mathbb{T}, m)$.

Lemma 7.18. (*Kolmogorov, 1941*) *Let $w \geq 0$, $w \in L^1_+$. Then $(z^n)_{n\in\mathbb{Z}}$ is a minimal sequence in $L^2(wdm)$ if and only if $\frac{1}{w} \in L^1(\mathbb{T})$.*

Proof. Due to biorthogonality, we have

$$\delta_{n,k} = (z^n, f_k)_{L^2(wdm)} = \int_{\mathbb{T}} z^n \bar{f}_k w dm,\ n, k \in \mathbb{Z}.$$

Hence we deduce that $\bar{f}_k w = \bar{z}^k$, $k \in \mathbb{Z}$, that is $f_k = \frac{z^k}{w}$, $k \in \mathbb{Z}$. Hence

$$f_k \in L^2(wdm) \text{ if and only if} \int_{\mathbb{T}} \frac{1}{w^2} w dm < \infty.$$

□

7.3 Riesz projection

See Figure 7.1 for a schematic illustration.

Let $\mathbb{P}, \mathbb{P}_+$ be as earlier and $\mathbb{P}_- = \text{span}\{e^{ikt} : k < 0\}$. Define the Riesz projection P_+ by

$$P_+ f = \sum_{k\geq 0} \hat{f}(k)e^{ikt},\ f \in \mathbb{P}.$$

Then

$$P_+ = P_{\mathbb{P}_+||\mathbb{P}_-}.$$

Let also

$$P_{m,n} f = \sum_{k=m}^{n} \hat{f}(k)e^{ikt},\ f \in \mathbb{P},\ m, n \in \mathbb{Z},\ m \leq n.$$

The following result gives the principle link between the problem of bases and the norm estimation of the Riesz projection.

Lemma 7.19. *Let $w \in L^1_+$. Then the following are equivalent.*

(i) $(z^k)_{k\in\mathbb{Z}}$ *is a nonsymmetric basis of* $L^2(wdm)$.

(ii) $\sup\limits_{n,m\in\mathbb{Z}} \|P_{m,n}\| < \infty$.

(iii) $(z^k)_{k\in\mathbb{Z}}$ *is a symmetric basis of* $L^2(wdm)$.

(iv) $\sup\limits_{n\in\mathbb{Z}} \|P_{-n,n}\| < \infty$.

(v) The Riesz projection P_+ *is continuous on* $L^2(wdm)$.

(vi) $\langle P_+, P_-\rangle > 0$ *(or* $\langle H^2_+, H^2_-\rangle > 0$, *where* $H^2_\pm = \text{clos}_{L^2(wdm)}\,\mathbb{P}_\pm$.

Proof. In view of Lemma 7.10 we get (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv). It is also clear that (ii) implies (iv). Using Lemma 7.14 and Corollary 7.15 we obtain (v) $\Leftrightarrow$ (vi). Next, we verify that (iv) implies (v). Pick $f \in \mathbb{P}$, then for $n = n(f)$ sufficiently large, we get (using the relation: $\widehat{z^{-n}f}(k) = \hat{f}(n+k)$), $P_+ f = z^n P_{-n,n} z^{-n} f$, so $\|P_+ f\| = \|P_{-n,n} z^{-n} f\| \leq \|P_{-n,n}\|\|f\|$ implies $\|P_+\| \leq \sup\limits_{n\geq 1} \|P_{-n,n}\|$. It remains to show that (v) implies (ii). Note that

$$P_{m,n} f = z^{n+1}(1 - P_+)z^{-(n+m+1)} P_+ z^m f,\quad f \in \mathbb{P}.$$

But then

$$\|P_{m,n} f\| = \|(1 - P_+)z^{-(n+m+1)} P_+ z^m f\| \leq \|P_+\|\|P_+ z^m f\| \leq \|P_+\|^2\|f\|$$

for all $f \in \mathbb{P}$, since $\|1 - P_+\| = \|P_+\|$, (by Corollary 7.15). Hence the result follows. $\square$

7.4 Harmonic conjugates

In order to get the desired characterization of exponential type bases in $L^2(\mu)$, we need a result of analytic type, namely, the so-called harmonic conjugation on $\mathbb{T}$ (or $\mathbb{D}$).

Theorem 7.20. *Let $u \in L^2(\mathbb{T})$ be a real valued function. Then there exist a unique real valued function $v \in L^2(\mathbb{T})$ such that $\hat{v}(0) = 0$ and $u + iv \in H^2$. The mapping $u \mapsto v$ is linear and continuous with $\|v\| \leq \|u\|$.*

Proof. Let $u = \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{int} \in L^2$. Then $\bar{u} = \sum_{n\in\mathbb{Z}} \bar{\hat{u}}(n)e^{-int}$. Since u is real valued, $\bar{u} = u \Leftrightarrow \overline{\hat{u}(n)} = \hat{u}(-n)$ for $n \in \mathbb{Z}$. Define

$$f = \hat{u}(0) + 2\sum_{n\geq 1} \hat{u}(n)z^n.$$

Then $f \in H^2$ and

$$\operatorname{Re} f = \frac{1}{2}(f + \bar{f}) = \hat{u}(0) + \sum_{n\geq 1} \hat{u}(n)e^{int} + \sum_{n\geq 1} \overline{\hat{u}(n)}e^{-int} = u.$$

Thus $v := \operatorname{Im} f$ satisfies the conclusion of the theorem. We now prove uniqueness. If $u + iv = u + iv_1 \in H^2$, then $v - v_1 \in H^2$. Since $v - v_1$ is real-valued, its complex conjugate also belongs to H^2, and therefore $v - v_1$ must be constant. Moreover,

$$v(0) - v_1(0) = \widehat{v}(0) - \widehat{v}_1(0) = 0$$

by the normalization $\widehat{v}(0) = \widehat{v}_1(0) = 0$. Hence $v = v_1$. Finally,

$$v = \operatorname{Im} f = \frac{f - \bar{f}}{2i} = \frac{1}{i}\Big(\sum_{n\geq 1} \hat{u}(n)e^{int} - \sum_{n\geq 1} \bar{\hat{u}}(n)e^{-int}\Big) = \frac{1}{i}\Big(\sum_{n>0} \hat{u}(n)e^{int} - \sum_{n<0} \hat{u}(n)e^{int}\Big).$$

The process $u \longmapsto v$ is linear and

$$\|v\|^2 = \sum_{k\neq 0} |\hat{u}(k)|^2 \leq \|u\|^2,$$

and if $\hat{u}(0) = 0$, then $\|u\| = \|v\|$. $\square$

Definition 7.21. The function v is called Harmonic conjugate of u. Let $v = \tilde{u}$. The mapping $H : L^2(\mathbb{T}) \to L^2(\mathbb{T}),\ u \longmapsto \tilde{u}$ is called the **Hilbert transform**.

7.5 Different formulas for $\tilde{u}$

(a) We can translate the above formula for $\tilde{u}$ in terms of Riesz projections

$$\tilde{u} = \frac{1}{i}(P_+ u - P_- u) - \frac{1}{i}\hat{u}(0).$$

In particular, if $\hat{u}(0) = 0$, then $\tilde{u} = \frac{1}{i}(P_+ u - P_- u)$. Also, we have

$$f = u + i\tilde{u} = 2P_+ u - \hat{u}(0).$$

(b) If u satisfies the hypotheses of the theorem, then $f = u + iv \in H^2$ and $u = \operatorname{Re} f$. Since f extends holomorphically to $\mathbb{D}$, so does $\operatorname{Re} f$. For $z \in \mathbb{D}$, we have

$$u(z) = \operatorname{Re}(f * P_z) = u * P_z.$$

Because the Poisson kernel satisfies $P_z(\zeta) = \operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right)$, it follows that $u(z) = \operatorname{Re} f_1(z)$, where

$$f_1(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} u(\zeta) dm(\zeta).$$

Note that $f_1 \in \operatorname{Hol}(\mathbb{D})$, $\operatorname{Re} f_1 = u$, and $f_1(0) = \int_{\mathbb{T}} u\, dm \in \mathbb{R}$. By uniqueness, we have $f = f_1$ and

$$\tilde{u} = \operatorname{Im} f = \operatorname{Im} f_1 = \int_{\mathbb{T}} \operatorname{Im}\left(\frac{\zeta + z}{\zeta - z}\right) u(\zeta)\, dm(\zeta) = \int_0^{2\pi} Q_r(\tau - t) u(e^{it}) \frac{dt}{2\pi},$$

where $z = re^{i\tau}$ and

$$Q_r(t) = \operatorname{Im}\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{2r \sin t}{1 - 2r\cos t + r^2}.$$

Auxiliary computation. For $|z| < 1$,

$$\frac{e^{it} + z}{e^{it} - z} = 1 + 2\sum_{n=1}^{\infty} z^n e^{-int}.$$

Therefore

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\,f(e^{it})\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{it})\,dt+\frac{2}{2\pi}\int_{-\pi}^{\pi}\sum_{n=1}^{\infty}z^n e^{-int}f(e^{it})\,dt\\ &= \widehat{f}(0)+2\sum_{n=1}^{\infty}\widehat{f}(n)\,z^n,\end{aligned}$$

which is a power series in z and hence defines a holomorphic function on $\mathbb{D}$ (see [9, p. 12]).

Remark 7.22. For $r\to 1$, $Q_r\sim\dfrac{\sin t}{1-\cos t}=\cot(t/2)$. In fact, one can show that

$$\tilde{u}(\tau)=(u*\cot(./2))(\tau)=\int_0^{2\pi}u(\tau-t)\cot\Big(t/2\Big)\frac{dt}{2\pi}$$

in the sense of Cauchy principle valued integral.

7.6 The Helson–Szegö theorem

Theorem 7.23. *Let μ be a finite Borel measure on $\mathbb{T}$. Then the following are equivalent.*

(i) The family $(z^n)_{n\in\mathbb{Z}}$ is a (symmetric or nonsymmetric) basis of $L^2(\mu)$.

(ii) The Riesz projection P_+ is bounded on $L^2(\mu)$.

(iii) The angle satisfies $\sin\langle P_+,P_-\rangle>0$.

(iv) $d\mu=|h|^2dm$, where $h\in H^2$ is an outer function such that $\operatorname{dist}\left(\frac{\bar{h}}{h},H^\infty\right)<1$.

(v) $d\mu=wdm$, where $w=e^{u+\tilde{v}}$ and u,v are real valued bounded functions and $\|v\|_\infty<\frac{\pi}{2}$ (condition (HS)).

The proof of the theorem will be given in several steps based on the following lemmas.

Lemma 7.24. *The mapping $j:H^2\times H^2\to H^1$, $(\phi,\psi)\longmapsto\phi\psi$ is continuous and symmetric. Moreover, $j(B^2\times B^2)=B^1$, where B^p is the unit ball in H^p.*

Proof. The continuity follows from the Cauchy Schwarz inequality $\|\phi\psi\|_1\le\|\phi\|_2\|\psi\|_2$. For surjectivity, let $f\in H^1$, then $f=\lambda BS[f]$. Write $\phi=\lambda BS[f]^{\frac{1}{2}}$ and $\psi=[f]^{\frac{1}{2}}$ then $\phi\psi\in H^2$. □

Lemma 7.25. *Let E be a subspace of the Banach space X, and $\Phi\in X^*$. Then*

$$\|\Phi|_E\|=\inf\{\|\Psi\|_{X^*}:\ \Psi=\Phi\ \textit{on}\ E\}=\inf\{\|\Phi+\alpha\|_{X^*}:\ \alpha\in X^*\ \textit{and}\ \alpha|_E=0\}$$

Proof. The inequality "$\leq$" is clear. For "$\geq$" apply Hahn-Banach's theorem. Let $\Psi' = \Phi|_E$. Then

$$\|\Psi\|_{X^*} = \sup_{x \in X} |\Psi(x)| \geq \|\Psi'\|_{X^*} = \sup_{x \in X} |\Psi'(x)| = \|\Phi|_E\|.$$

By Hahn-Banach's theorem, there exists $\Psi' \in X^*$ such that $\|\Phi|_E\| = \|\Psi'\|_{X^*}$, and hence the result follows. □

Lemma 7.26. *Let $f \in H^1$ and suppose that $f(\mathbb{T}) \subset A \subset \mathbb{C}$. Then $f(\mathbb{D}) \subset \operatorname{conv} A$) (the closed convex hull of A).*

Proof. Observe that for $z = rw \in \mathbb{D}, |w| = 1$ we have $f(z) = P_z * f = \int\limits_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} f(\zeta) d\zeta \in$ conv(A). However, conv$(A) = \cap H$ where the intersection is taken over all the half-planes: $H = \{z \in \mathbb{C} : \operatorname{Re}(az + b) \geq 0\}$ containing A, $a, b \in \mathbb{C}$. Since $P_r > 0$ and $\int_{\mathbb{T}} P_r dm(\xi) = 1$, we see that the condition $\operatorname{Re}(af(\zeta + b) \geq 0)$ for a.e. $\zeta \in \mathbb{T}$ as $f(\zeta) \in A \subset H \implies \operatorname{Re}(af(z) + b) \geq 0 \implies f(z) \in \operatorname{conv}(A)$ □

Lemma 7.27. (*V. Smirnov, A. Kolmogorov*) *Let $v \in L^\infty(\mathbb{T})$ be a real valued function then $e^{\lambda \tilde{v}} \in L^1(\mathbb{T})$ if $\lambda \|v\|_\infty < \frac{\pi}{2}$.*

Proof. It is enough to prove that $\|u\|_\infty < \pi/2$ implies $e^{\tilde{u}} \in L^1$. Set

$$f = e^{-i(u + i\tilde{u})}.$$

This is well defined on $\mathbb{D}$ because $u + i\tilde{u} \in H^2$. Moreover, $|f| = e^{\tilde{u}}$ and $|\arg f| = |u| < \pi(1 - \varepsilon)/2$ for some $\varepsilon > 0$ on $\mathbb{T}$, and hence on $\mathbb{D}$ by Lemma 7.26. The same argument as in Theorem 6.9 therefore shows that $f \in H^1$, so $|f| = e^{\tilde{u}} \in L^1(\mathbb{T})$. □

Proof. **Implication (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) of Helson-Szegö theorem.**

Recall that we may restrict to $d\mu = w dm$, $w \in L^1_+(\mathbb{T})$. By Lemma 7.19 we get the equivalence of (i),(ii) and (iii).

We next prove the equivalence of (i), (ii), and (iv); see Figure 7.2. If the sequence $(z^n)_{n \in \mathbb{Z}}$ is a basis, then Banach's theorem (Theorem 7.8) together with Kolmogorov's lemma (Lemma 7.18) gives $1/w \in L^1$, and hence $\log w \in L^1$. (Indeed, the latter conclusion can also be justified directly from $\bar{z} \notin H^2(\mu)$.) Therefore there exists an outer function $h \in H^2$ such that $|h|^2 = w$. Thus,

$$(f, g)_{L^2(\mu)} = \int_{\mathbb{T}} f \bar{g} w dm = \int_{\mathbb{T}} f h \bar{g} h \frac{\bar{h} h}{h^2} dm = \int_{\mathbb{T}} (fh)(\bar{g}h) \frac{\bar{h}}{h} dm = \int_{\mathbb{T}} FG \frac{\bar{h}}{h} dm$$

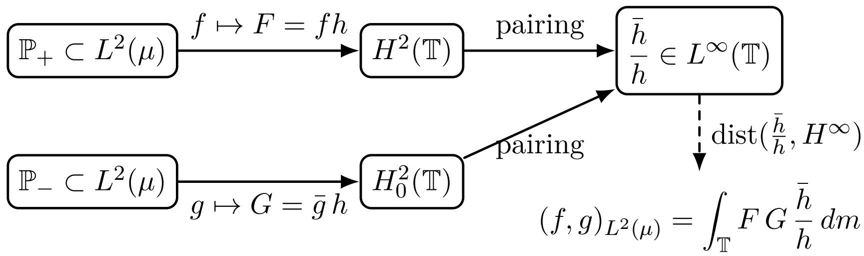

Figure 7.2: A schematic of the identifications in the proof of the Helson–Szegő theorem: multiplication by an outer h transfers the weighted pairing on $L^2(\mu)$ to an unweighted pairing on $\mathbb{T}$ with symbol $\bar h/h$.

for all $f \in \mathbb{P}_+$ and $g \in \mathbb{P}_-$ and therefore,

$$\|f\|^2_{L^2(\mu)} = \int |fh|^2 dm = \|F\|^2_{L^2(\mathbb{T})}, \;\; \|g\|^2_{L^2(\mu)} = \|G\|^2_{L^2(\mathbb{T})}.$$

Clearly $F = fh \in H^2$, since $\bar g \in \mathbb{P}^0_+$, we get $G \in H^2_0$. By definition of outer function, it follows that $\overline{\operatorname{span}}\{F = fh : f \in \mathbb{P}_+\} = H^2$, and also $A := \{F = fh : f \in \mathbb{P}_+, \|F\| \le 1\}$ is dense in the unit ball B^2 of H^2. For the same reason, we see that $B := \{G = \bar g h : g \in \mathbb{P}_-, \|G\| \le 1\}$ is dense in $B^2 \cap H^2_0$. We deduce that

$$\cos\langle \mathbb{P}_+, \mathbb{P}_-\rangle_{L^2(\mu)} = \sup\{|(f,g)| : f \in \mathbb{P}_+, g \in \mathbb{P}_- \, \|f\|^2_L(\mu) \le 1, \, \|g\|^2_L(\mu) \le 1\}$$

$$= \sup\left\{\Big|\int_{\mathbb{T}} FG\frac{\bar h}{h} dm\Big| : F \in A, \, G \in B\right\}.$$

Set $\Phi(k) = \int\limits_{\mathbb{T}} k(\frac{\bar h}{h})dm$, $k \in L^1(\mathbb{T})$. As $\bar h/h \in L^\infty(\mathbb{T})$, we get $\Phi \in (L^1(\mathbb{T}))^*$. By (Lemma 7.24), we see that the angle $\langle \mathbb{P}_+, \mathbb{P}_-\rangle = \|\Phi|_{H^1_0}\|$, and by means of (Lemma 7.25), we can express it in terms of h:

$$\cos\langle \mathbb{P}_+, \mathbb{P}_-\rangle_{L^2(\mu)} = \|\Phi|_{H^1_0}\| = \operatorname*{dist}_{L^\infty(\mathbb{T})}\left(\frac{\bar h}{h}, (H^1_0)^\perp\right) = \operatorname*{dist}_{L^\infty(\mathbb{T})}\left(\frac{\bar h}{h}, H^\infty\right).$$

The last equality is the consequence of the relation

$$(H^1_0)^\perp = \{g \in L^\infty : \int_{\mathbb{T}} gf dm = 0 \text{ for all } f \in H^1_0\} = H^\infty.$$

Next, we conclude that $\cos\langle \mathbb{P}_+, \mathbb{P}_-\rangle < 1$ if and only if $\log w \in L^1$, $w = |h|^2$ for an outer function $h \in H^2$ satisfying $\mathrm{dist}_{L^\infty(\mathbb{T})}(\frac{\bar h}{h}, H^\infty) < 1$, that is (i) and (ii) are equivalent to (iv).
Proof of implication (iv) $\implies$ (v):(See **Fig 2**)

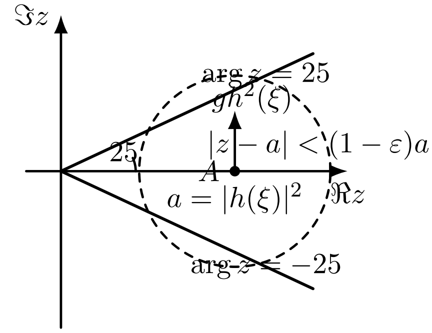

Figure 7.3: The geometric constraint $|a - gh^2(\xi)| < (1-\varepsilon)a$ forces $gh^2(\mathbb{T})$ to lie in a sector $A = \{z : |\arg z| < \alpha\}$, with $\sin\alpha = 1 - \varepsilon$.

See Figure 7.3 for a schematic illustration.

Suppose that

$$\operatorname*{dist}_{L^\infty(\mathbb{T})}\left(\frac{\bar{h}}{h}, H^\infty\right) < 1,$$

where h is an outer function and $|h|^2 = w$. Then there exists $g \in H^\infty$ such that

$$\left\|\frac{\bar{h}}{h} - g\right\|_\infty < 1.$$

Equivalently, for some $\varepsilon > 0$ we have

$$\left|\frac{\bar{h}}{h} - g\right| < 1 - \varepsilon \qquad \text{a.e. on } \mathbb{T}.$$

Multiplying by $|h|^2$, we obtain

$$\left||h|^2 - gh^2\right| < (1-\varepsilon)|h|^2 \qquad \text{a.e. on } \mathbb{T}.$$

Setting $a = |h(\xi)|^2 > 0$ for $\xi \in \mathbb{T}$, this becomes

$$|a - gh^2| < (1-\varepsilon)a.$$

Geometrically, if $\alpha \in (0, \pi/2)$ is chosen so that $\sin\alpha = 1 - \varepsilon$ and $A = \{z : |\arg z| < \alpha\}$, then $gh^2(\mathbb{T}) \subset A$; see Figure 7.3.

From (Lemma 7.26) we get $gh^2(\mathbb{D}) \subset A$, so $\log gh^2$ is analytic in $\mathbb{D}$. We set $v = -\operatorname{Im}\log gh^2 = -\arg gh^2$ and get $|\tilde{v}| = \operatorname{Re}\log gh^2 + c = \log|gh|^2 + c$, where c has to be chosen such that $\tilde{v}(0) = 0$. We obtain $\log gh^2 = \tilde{v} - iv - c$ and $gh^2 = e^{\tilde{v}-iv-c}$ on $\mathbb{T}$, we have $|\frac{\bar{h}}{h} - g| < 1 - \epsilon$, which implies that $|1 - |g|| < 1 - \epsilon$, hence $\epsilon \le |g| \le 2 - \epsilon$. Finally,

$|h|^2 = \dfrac{e^{\tilde{v}-c}}{|g|} = e^{u+\tilde{v}}$, where $u = -\log|g| - c \in L^\infty(\mathbb{T})$ and $\|v\|_\infty < \frac{\pi}{2}$.

Proof of implication (v) implies (iv):

Let $wdm = e^{u+\tilde{v}}dm$, where $u, v \in L^\infty(\mathbb{T})$ are real valued and $\|v\|_\infty < \frac{\pi}{2}$. Since $\log w = u + \tilde{v} \in L^1(\mathbb{T})$, Lemma 7.27 implies that $w \in L^1(\mathbb{T})$. Hence there exists an outer function $h \in H^2$ such that $|h|^2 = w$. Therefore

$$\log|h|^2 = u + \tilde{v}$$

and

$$\log h^2 = u + \tilde{v} + i(u+\tilde{v})^\sim = u + \tilde{v} + i(\tilde{u} - v + c)$$

for some constant $c \in \mathbb{R}$. Setting $g = e^{-(u+i\tilde{u})-ic}$, we obtain a bounded holomorphic function $g \in H^\infty$ because $|g| = e^{-u}$. Moreover,

$$\frac{h}{\bar{h}}g = \frac{h^2}{|h|^2}g = \exp(i(\tilde{u} - v + c) - u - i\tilde{u} - ic) = \exp(-u - iv),$$

where $\|v\|_\infty < \frac{\pi}{2}$. This gives the following estimates on $\mathbb{T}$.

$$e^{-\|u\|_\infty} \le \left|\frac{h}{\bar{h}}g\right| \le e^{\|u\|_\infty}, \quad \left|\arg(\frac{h}{\bar{h}})g\right| = |v| < \pi\frac{(1-\epsilon)}{2}.$$

(cf. Figure 2). The value of $(\frac{h}{\bar{h}})g$ thus belongs to

$$\mathcal{D} := \left\{z \in \mathbb{C} : \; e^{-\|u\|_\infty} \le |z| \le e^{\|u\|_\infty}, \quad |\arg z| < \pi\frac{(1-\epsilon)}{2}\right\}.$$

For λ sufficiently big and some $\delta > 0$ we have $B(\lambda, (1-\delta)\lambda) \supset \operatorname{clos}\mathcal{D}$ or $\lambda^{-1}B(\lambda,(1-\delta)\lambda) = B(1, 1-\delta) \supset \lambda^{-1}\operatorname{clos}\mathcal{D}$. Then $\lambda^{-1}\frac{h}{\bar{h}}g \in B(1, 1-\delta)$ a.e. on $\mathbb{T}$. In other words, $|\lambda^{-1}(\frac{h}{\bar{h}})g - 1| < 1-\delta$ a.e. on $\mathbb{T}$, and $|\lambda^{-1}g - (\frac{\bar{h}}{h})| < 1-\delta$ a.e. $\mathbb{T}$. As $g \in H^\infty$, this gives $\operatorname{dist}_{L^\infty(\mathbb{T})}(\frac{\bar{h}}{h}, H^\infty) < 1$. □

7.7 An example

Let $\omega(e^{it}) = |t|^\alpha$, $t \in (-\pi, \pi)$, $\alpha \in \mathbb{R}$. Then for $\alpha \ge 1$ we have $1/\omega \notin L^1(\mathbb{T})$ and $(e^{int})_{n\in\mathbb{Z}}$ cannot be uniformly minimal in view of Lemma 7.18. For $\alpha \le -1$, $\omega \notin L^1$. Thus, the only interesting case is $|\alpha| < 1$.

First note that if the quotient ω_1/ω_2 and ω_2/ω_1 are bounded, then the sequence $(e^{int})_{n\in\mathbb{Z}}$ is a basis of $L^2(\omega_1)$ if and only if it is one of $L^2(\omega_2)$. $\Big[$ $|\frac{w_1}{w_2}| < K$ and $|\frac{w_2}{w_1}| < K_1$. By the

Lemma 7.18, $(e^{int})_{n\in\mathbb{Z}}$ is a basis of $L^2(w_1) \Leftrightarrow \frac{1}{w_1} \in L^1$. Now

$$\int |\frac{1}{w_2}| \leq \int |\frac{K}{w_1}| = K \int \frac{1}{|w_1|} < \infty \Rightarrow \frac{1}{w_2} \in L^1 \Leftrightarrow$$

$(e^{int})_{n\in\mathbb{Z}}$ is a basis of $L^2(w_2)$ by Lemma 7.18. Similarly the other case follows.]

The identity map $f \longmapsto f$ is an isomorphism from $L^2(\omega_1)$ to $L^2(\omega_2)$.

Next, let $\omega_1 = \omega$ and $\omega_2 = (1-e^{it})^\alpha$. Then

$$\log \omega_2 = \log |1-e^{it}|^\alpha = \alpha \operatorname{Re} \arg(1-e^{it}) := u.$$

Necessarily, we get

$$\begin{aligned}
\tilde{u}(t) &= \alpha \arg(1-e^{it}) = \alpha \arg(e^{it/2}(e^{-it/2}-e^{it/2}) \\
&= \alpha \arg(e^{it/2}(-2i\sin t/2). \\
&= \begin{cases} \alpha(t/2-\pi/2) & \text{if } t>0 \\ \alpha(\pi/2-t/2) & \text{if } t<0. \end{cases}
\end{aligned}$$

We deduce that $\|\tilde{u}\|_\infty = |\alpha|\frac{\pi}{2} < \frac{\pi}{2}$ if $|\alpha|<1$. Hence $(e^{int})_{n\in\mathbb{Z}}$ is a basis in $L^2(|t|^\alpha dt) \Leftrightarrow$ $|\alpha|<1$.

Chapter 8

Transfer to the upper half-plane

We transport the disk theory to the upper half-plane via the Cayley transform. After constructing an explicit isometric correspondence between $L^p(\mathbb{T})$ and $L^p(\mathbb{R})$, we develop the Hardy spaces $H^p(\mathbb{C}_+)$ from both the boundary-value and Fourier-transform viewpoints (Paley–Wiener). We discuss canonical factorization, invariant subspaces, and the duality between translations and multiplications by characters, emphasizing the parallelism with the disk model.

Learning objectives.

- Transfer the disk model to $\mathbb{C}_+$ through the Cayley transform and boundary Jacobian.
- Connect $H^2(\mathbb{C}_+)$ with the Fourier transform and Paley–Wiener theory.
- Translate inner functions, factorization, and invariant subspaces from the disk to the upper half-plane.

Key ideas.

- The upper half-plane is not merely another model: it makes translation-invariance and Fourier support visible.
- Outer/inner factorization survives conformal transport, but its formulas acquire a distinctly real-variable flavour.
- The duality between multiplication by characters and translation links Hardy-space methods to spectral analysis.

Example 8.1 (A basic inner function on $\mathbb{C}_+$)**.** For $a \geq 0$, the function

$$I_a(z) = e^{iaz}, \qquad z \in \mathbb{C}_+,$$

is bounded and analytic on $\mathbb{C}_+$, satisfies $|I_a(x)| = 1$ for almost every $x \in \mathbb{R}$, and hence is inner. This family already reflects the translation/Fourier flavour that is natural in the half-plane model.

In this chapter we outline the Hardy-space theory on the upper half-plane and on the real line. We focus on the principal structural results: the isometric correspondence between Hardy spaces on the disk and on the half-plane, canonical factorization, the Fourier-transform description given by the Paley–Wiener theorem, and invariant subspaces.

8.1 A unitary mapping from $L^p(\mathbb{T})$ to $L^p(\mathbb{R})$

Let $\omega : \mathbb{D} \to \mathbb{C}$ be the Cayley transform,

$$\omega(z) = i\frac{1+z}{1-z},$$

which maps the disk $\mathbb{D}$ conformally onto the upper half-plane

$$\mathbb{C}_+ = \{\xi \in \mathbb{C} : \text{Im } \xi > 0\}.$$

Its boundary restriction $\omega|_{\mathbb{T}}$ is a one-to-one correspondence between $\mathbb{T} \setminus \{1\}$ and $\mathbb{R}$. The inverse map is

$$\omega^{-1}(x) = \frac{x-i}{x+i}, \qquad x \in \mathbb{R},$$

and its Jacobian satisfies $|J(x)| = \dfrac{2}{1+x^2}$. Hence the mapping

$$U = U_p : L^p(\mathbb{T}) \to L^p(\mathbb{R})$$

$$U_p f(x) = \left(\frac{1}{\pi(x+i)^2}\right)^{1/p} f(\omega^{-1}(x)),\ x \in \mathbb{R},$$

is an isomorphism (unitary for $p = 2$) of the space $L^p(\mathbb{T})$ onto $L^p(\mathbb{R})$.

First, we give three descriptions of the image under U of the Hardy-space $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$, then pass to arbitrary p, $1 \leq p \leq \infty$. Consequently, $U_p H^p(\mathbb{T})$ is a closed subspace of $L^p(\mathbb{R})$.

8.2 Cauchy kernel and Fourier transform

The first description of $U_2H^2(\mathbb{T})$ is straightforward.

Lemma 8.2.
$$U_2H^2(\mathbb{T}) = \overline{\text{span}}_{L^2(\mathbb{R})}\left\{\frac{1}{x-\bar{\mu}} : \text{Im } \mu > 0\right\}.$$

To prove this we first need the following proposition:

Proposition 8.3. $H^2(\mathbb{D}) = \overline{\text{span}}\left\{c_\lambda = \frac{1}{1-\overline{\lambda}z} : \lambda \in \mathbb{D}\right\}$

Proof. From Corollary 5.10, for each $\lambda \in \mathbb{D}$ the evaluation map $\varphi_\lambda(f) = f(\lambda)$ is a bounded linear functional on H^2. By the Riesz representation theorem, there exists a unique $c_\lambda \in H^2$ such that
$$\varphi_\lambda(f) = f(\lambda) = \langle f, c_\lambda\rangle \qquad (f \in H^2).$$

We now compute c_λ. For $|\lambda| < 1$ we have
$$\frac{1}{1-\overline{\lambda}z} = \sum_{n\geq 0} \overline{\lambda}^n z^n,$$

and $(\overline{\lambda}^n)_{n\geq 0} \in \ell^2(\mathbb{N}_0)$, so this function belongs to H^2. Moreover,
$$\left\|\frac{1}{1-\overline{\lambda}z}\right\|_{H^2}^2 = \sum_{n\geq 0} |\lambda|^{2n} = \frac{1}{1-|\lambda|^2} < \infty.$$

If $f(z) = \sum_{n\geq 0} a_n z^n \in H^2$, then
$$\left\langle f, \frac{1}{1-\overline{\lambda}z}\right\rangle = \sum_{n\geq 0} a_n \lambda^n = f(\lambda).$$

By uniqueness in the Riesz representation theorem, it follows that
$$c_\lambda(z) = \frac{1}{1-\overline{\lambda}z}.$$

Consequently,
$$\|c_\lambda\|^2 = \langle c_\lambda, c_\lambda\rangle = c_\lambda(\lambda) = \frac{1}{1-|\lambda|^2}.$$

The function c_λ is called the **Cauchy kernel** or **Szegő kernel**, and H^2 is a **reproducing kernel Hilbert space**.

It is straightforward to check that the family $D = \{c_\lambda : \lambda \in \mathbb{D}\}$ is linearly independent. Moreover, if $f \in H^2$ is orthogonal to c_λ for every $\lambda \in \mathbb{D}$, then $f(\lambda) = \langle f, c_\lambda \rangle = 0$ for all $\lambda \in \mathbb{D}$, so $f = 0$. Hence D is dense in H^2 (equivalently, $D^\perp = \{0\}$). $\square$

Proof. [Proof of Lemma 8.2] Since $H^2(\mathbb{T}) = \overline{\text{span}}_{L^2(\mathbb{T})}\left\{\frac{1}{1-\bar{\lambda}z} : |\lambda| < 1\right\}$, and U_2 is an isometry, we have

$$H^2(\mathbb{T}) = \overline{\text{span}}_{L^2(\mathbb{T})}\left\{U_2(1-\bar{\lambda}z)^{-1} = \frac{C_\lambda}{z - \overline{\omega(\lambda)}} : \lambda \in \mathbb{D}\right\} = \overline{\text{span}}\{\frac{1}{z-\overline{\mu}} : \text{Im}\,\mu > 0\}.$$

Consequently, $\mu = \omega(\lambda)$ runs over the entire upper half-plane $\mathbb{C}_+$. $\square$

Now, we recall that Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$,

$$\mathcal{F}(f)(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ixz}dx,$$

$$\mathcal{F}^{-1}(f)(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{ixz}dx$$

are unitary mapping of $L^2(\mathbb{R})$ onto itself.

Lemma 8.4. *$U_2H^2 = \mathcal{F}^{-1}L^2(\mathbb{R}_+)$, where $L^2(\mathbb{R}_+) = \{f \in L^2(\mathbb{R}) : f = 0$ on $(-\infty, 0)\}$.*

Proof. Compute the inverse Fourier transform of the function $\chi_{\mathbb{R}_+}e^{i\lambda x} \in L^2(\mathbb{R}_+)$, where $\text{Im}\,\lambda > 0$:

$$\mathcal{F}^{-1}(\chi_{\mathbb{R}_+}e^{i\lambda x}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \chi_{\mathbb{R}_+}e^{i\lambda x}e^{ixz}dx = \frac{1}{\sqrt{2\pi}}\frac{1}{i(z+\lambda)}\left[ix(z+\lambda)\right]_{x=0}^{\infty} = \frac{i}{\sqrt{2\pi}}\frac{1}{z-(-\lambda)},$$

where $-\lambda = \mu$ runs, again, over the entire half-plane $\mathbb{C}_+$. Since $\mathcal{F}^{-1}$ is an isometry, Lemma 8.4 reduces to the proof of the following identity:

$$L^2(\mathbb{R}_+) = \overline{\text{span}}\{\chi_{\mathbb{R}_+}e^{i\lambda x}\ \text{Im}\,\lambda > 0\}.$$

The equality follows from the injectivity (classical Fourier uniqueness theorem) of the

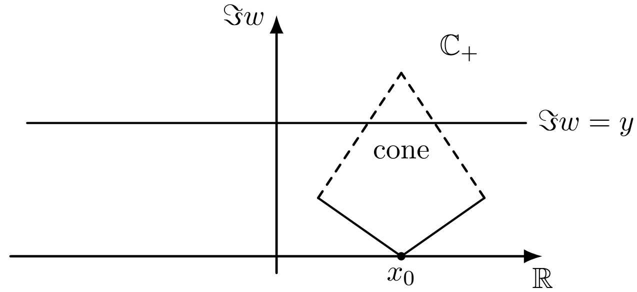

Figure 8.1: Typical regions in $\mathbb{C}_+$: horizontal slices $\Im w = y$ used in the $H^p(\mathbb{C}_+)$ norm, and a non-tangential approach cone at $x_0 \in \mathbb{R}$.

Fourier transform $\mathcal{F}$. Let $f \in L^2(\mathbb{R}_+)$ and suppose that $f \perp \chi_{\mathbb{R}_+} e^{i\lambda x}$ for all λ with Im $\lambda > 0$.

$$\begin{aligned}
&\int_{\mathbb{R}} f(x)\chi_{\mathbb{R}_+} e^{-\lambda x} dx = 0 \\
&\implies \int_{\mathbb{R}} f(x)\chi_{\mathbb{R}_+} e^{-x} e^{-iyx} dx = 0 \text{ (putting } \lambda = y + i) \\
&\implies \mathcal{F}(f\chi_{\mathbb{R}_+} e^{-x})(y) = 0 \ (\forall y \in \mathbb{R}) \\
&\implies f\,\chi_{\mathbb{R}_+} e^{-x} = 0 \ \text{ a.e. on } \mathbb{R} \text{ [since } \widehat{f} = 0 \implies f = 0] \\
&\implies f = 0
\end{aligned}$$

□

8.3 The Hardy space $H^p_+ = H^p(\mathbb{C}_+)$

Section roadmap.

- Define the half-plane Hardy spaces through boundary control and Poisson extension.
- Transport the disk theory via the Cayley transform while keeping track of norms and boundary measures.
- Then reinterpret the resulting spaces through Fourier-analytic tools that are special to $\mathbb{C}_+$.

See Figure 8.1 for a schematic illustration.

We now transfer the theory from the real line $\mathbb{R}$ to the upper half-plane $\mathbb{C}_+$. We identify the subspace $U_p H^p \subset L^p(\mathbb{R})$ with the space of boundary values of a corresponding holomorphic Hardy space in the half-plane $\mathbb{C}_+$. Note that $\omega^{-1}(z) = \frac{z-i}{z+i}$ is a conformal mapping from $\mathbb{C}_+$ to $\mathbb{D}$.

Hence the same formula as above, $U_p : H^p(\mathbb{C}^+) \to H^p(\mathbb{D})$

$$U_p f(z) = \left(\frac{1}{\pi(z+i)}\right)^{1/p} f(\omega^{-1}(z)), \text{ Im } z > 0$$

defines a holomorphic function in $\mathbb{C}_+$ for all $f \in H^p(\mathbb{C}_+)$.

Moreover, ω^{-1} is still conformal at the boundary points $r \in \mathbb{R}$ and transfers a Stolz angle in $\mathbb{C}_+$, $\{x+iy : |x-r| < cy\}$, into a Stolz angle in $\mathbb{D}$. Now, Fatou's theorem implies that the functions $U_p f$, $f \in H^p(\mathbb{D})$, have non-tangential boundary limits $(U_p(f))_{\mathbb{R}}$ a.e. on $\mathbb{R}$, $U_p(f_{\mathbb{T}}) = (U_p f)_{\mathbb{R}}$. Hence in order to get another characterization of $U_p H^p(\mathbb{T})$, it remains to describe $U_p H^p(\mathbb{D})$ in intrinsic terms as a subset of $\text{Hol}(\mathbb{C}_+)$. This is done in the next theorem. But, first we define Hardy classes on $\mathbb{C}_+$.

Definition 8.5. Hardy space $H^p_+ = H^p(\mathbb{C}_+)$, $0 < p \leq \infty$, is the class of functions $g \in \text{Hol}(\mathbb{C}_+)$ such that

$$\|g\|_{H^p_+} = \sup_{y>0} \left(\int_{\mathbb{R}} |g(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty,$$

with the usual modification for $p = \infty$. In order to compare $H^p(\mathbb{C}_+)$ with $U_p H^p(\mathbb{D})$, we need the following simple result.

Lemma 8.6. *(i) Let γ be an arbitrary circle in $\overline{\mathbb{D}}$. Then*

$$\int_\gamma |f(z)|^p |dz| \leq 2 \int_{\mathbb{T}} |f(z)|^p |dz|$$

for all $f \in H^p(\mathbb{D})$, $1 \leq p < \infty$, here $|dz|$ stands for the arc length measure.

(ii) Let $g \in H^p(\mathbb{C}_+)$, $1 \leq p < \infty$ and $z \in \mathbb{C}_+$, then

$$|g(z)| \leq \left(\frac{2}{\pi \text{ Im } z}\right)^{\frac{1}{p}} \|g\|_{H^p_+}.$$

Proof. (i) We first treat the case $p = 1$. For $u \in L^1(\mu)$, let u_* denote the harmonic extension of u to the unit disk:

$$u_*(z) = \int_{\mathbb{T}} u(\zeta) \frac{1-|z|^2}{|\zeta - z|^2} dm(\zeta), \qquad z \in \mathbb{D}.$$

We claim that the map $u \mapsto u_*|_\gamma$ is a bounded operator from $L^1(\mathbb{T})$ to $L^1(\gamma)$ with norm at most 4π. Indeed,

$$\begin{aligned}\int_\gamma |u_*(z)||dz| &\leq \int_\gamma |u(\zeta)|\frac{1-|z|^2}{|\zeta-z|^2}dm(\zeta)|dz| \\ &= \int_{\mathbb{T}} |u(\zeta)|\Big(\int_\gamma \frac{1-|z|^2}{|\zeta-z|^2}|dz|\Big)dm(\zeta) \\ &= 2\pi r\int_{\mathbb{T}} |u(\zeta)|\frac{1-|c|^2}{|\zeta-c|^2}dm(\zeta),\end{aligned}$$

where $\gamma = \gamma(c, r)$. In the last step we used the mean-value theorem for harmonic functions, applied to the Poisson kernel

$$P_z(\zeta) = \mathrm{Re}\Big(\frac{z+\zeta}{z-\zeta}\Big).$$

Since $2\pi\, dm(z) = |dz|$ on $\mathbb{T}$, $r \leq 1-|c|$, and

$$\frac{1-|c|^2}{|\zeta-c|^2} \leq \frac{1+|c|}{1-|c|} \leq \frac{2}{1-|c|},$$

we obtain the desired inequality. For arbitrary p with $1 \leq p < \infty$, Hölder's inequality gives $|u_*|^p \leq (|u|^p)_*$, and the conclusion follows.

(ii) Using the mean-value theorem in the disk for

$$D = \{x+iy : \ |\lambda - (x+iy)| < \mathrm{Im}\,\lambda\},$$

Hölder's inequality, and the standard "rolling disk" argument, we obtain

$$\begin{aligned}|g(\lambda)| &= \frac{1}{\pi(\mathrm{Im}\,\lambda)^2}\int_D |g(z)|\,dx\,dy \\ &\leq \frac{1}{\pi(\mathrm{Im}\,\lambda)^2}\Big(\int_D |g(z)|^p\,dx\,dy\Big)^{1/p}\Big(\int_D 1\,dx\,dy\Big)^{1/q} \\ &\leq \Big(\frac{1}{\pi(\mathrm{Im}\,\lambda)^2}\Big)\Big(\int_D |g(z)|^p\,dx\,dy\Big)^{1/p}\Big(\pi(\mathrm{Im}\,\lambda)^2\Big)^{1/q} \\ &\leq \Big(\frac{1}{\pi\ \mathrm{Im}\,\lambda}\Big)^{2(1-\frac{1}{q})}\Big(\int_0^{2\,\mathrm{Im}\,\lambda}\int_{\mathbb{R}} |g(x+iy)|^p\,dx\,dy\Big)^{1/p} \\ &\leq \Big(\frac{2}{\pi\ \mathrm{Im}\,\lambda}\Big)^{1/p}\|g\|_{H^p_+}.\end{aligned}$$

□

The following theorem is one of the main results of this section.

Theorem 8.7. *Let $1 \le p \le \infty$. Then $U_p H^p(\mathbb{D}) = H^p(\mathbb{C}_+)$.*

Proof. Let $\omega : \mathbb{D} \to \mathbb{C}_+$ be the Cayley transform

$$\omega(z) = i\,\frac{1+z}{1-z}, \qquad \omega^{-1}(w) = \frac{w-i}{w+i}.$$

For $1 \le p < \infty$ define, for $w \in \mathbb{C}_+$,

$$(U_p f)(w) := \left(\frac{1}{\pi(w+i)^2}\right)^{1/p} f(\omega^{-1}(w)),$$

where the holomorphic branch of $(w+i)^{-2/p}$ is taken on $\mathbb{C}_+$ (note that $w+i$ never vanishes there). For $p=\infty$ we drop the prefactor and set $(U_\infty f)(w) = f(\omega^{-1}(w))$. The map U_p sends holomorphic functions on $\mathbb{D}$ to holomorphic functions on $\mathbb{C}_+$.

On the boundary one has, for almost every $x \in \mathbb{R}$, the change of variables $\zeta = \omega^{-1}(x) \in \mathbb{T} \setminus \{1\}$ with Jacobian $|d\zeta| = \frac{2}{1+x^2}\,dx$. Consequently,

$$\int_{\mathbb{R}} |U_p f(x)|^p\,dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{|f(\omega^{-1}(x))|^p}{|x+i|^2}\,dx = \int_{\mathbb{T}} |f(\zeta)|^p\,dm(\zeta),$$

and similarly for $p=\infty$ one obtains $\|U_\infty f\|_{L^\infty(\mathbb{R})} = \|f\|_{L^\infty(\mathbb{T})}$.

For $y>0$, the horizontal line $\mathbb{R}+iy$ is mapped by ω^{-1} to a circle $C_y \subset \mathbb{D}$ tangent to $\mathbb{T}$ at 1 (see Figure 8.2). The same change-of-variables computation yields

$$\int_{\mathbb{R}} |U_p f(x+iy)|^p\,dx = \int_{\mathbb{T}} |f(\varphi_y(\zeta))|^p\,dm(\zeta),$$

where φ_y is a disk automorphism (depending on y) parametrizing C_y. In particular, taking the supremum over $y>0$ shows that $f \in H^p(\mathbb{D})$ implies $U_p f \in H^p(\mathbb{C}_+)$ and

$$\|U_p f\|_{H^p(\mathbb{C}_+)} = \|f\|_{H^p(\mathbb{D})}.$$

Finally, the inverse transform is given by

$$(V_p g)(z) := \left(\pi(\omega(z)+i)^2\right)^{1/p} g(\omega(z)), \qquad z \in \mathbb{D} \quad (1 \le p < \infty),$$

and $V_\infty g = g \circ \omega$. One checks that $V_p \circ U_p = \mathrm{Id}$ and $U_p \circ V_p = \mathrm{Id}$ on boundary L^p spaces and hence on Hardy spaces. Therefore $U_p H^p(\mathbb{D}) = H^p(\mathbb{C}_+)$. □

See Figure 8.2 for a schematic illustration.

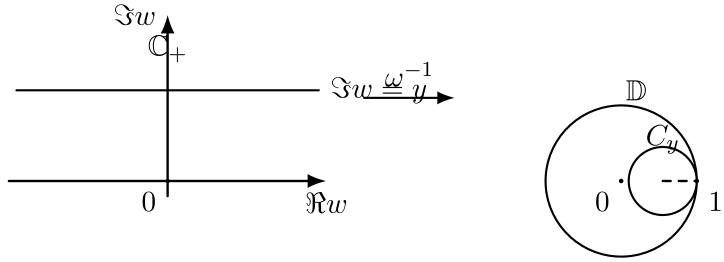

Figure 8.2: Under the Cayley transform, a horizontal line $\Im w = y$ in $\mathbb{C}_+$ corresponds to a circle $C_y \subset \mathbb{D}$ tangent to $\mathbb{T}$ at 1 (schematic).

Theorem 8.8. (*R. Paley and N. Wiener, 1934*)

$$H^p(\mathbb{C}_+) = \mathcal{F}^{-1}L^2(\mathbb{R}_+)$$

Proof. This is immediate from Lemma 8.4 and Theorem 8.7. □

8.4 Canonical factorization and other properties

The following properties are straightforward consequences of the change of variables from Section 8.1, Theorem 8.7, and the corresponding facts from H^p theory in the disk $\mathbb{D}$.

Corollary 8.9. (*Poisson formula*) *If* $f \in H^p(\mathbb{C}_+)$, $1 \leq p \leq \infty$, *then*

$$f(x+iy) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2} f(t)dt,\ y > 0.$$

Proof. $f \in H^P(\mathbb{C}_+) \implies$ there exists $g \in H^p(\mathbb{D})$ such that $U_p g(z) = f(z), z \in \mathbb{C}_+ \implies f(z) = (\frac{1}{\pi(z+i)})^{1/p} g(\frac{z-i}{z+i}), z \in \mathbb{C}_+$. Next put $w = \frac{z-i}{z+i} \in \mathbb{D}$ for $z \in \mathbb{C}_+$; then $(\frac{1}{z+i})^2 = (\frac{1-w}{2i})^2$ hence $f(z)$ can be re written as

$$\begin{aligned} f(z) &= \left(\frac{1}{\pi}(\frac{1-w}{2i})^2\right)^{1/p} g(w) \text{ for } z \in \mathbb{C}_+ \text{ and } w \in \mathbb{D}. \\ &= h(w) \in H^p(\mathbb{D}) \ \left[\text{since} \left(\frac{1}{\pi}(\frac{1-w}{2i})^2\right)^{1/p} \text{ is bounded on } \mathbb{D} \text{ and } g \in H^p(\mathbb{D})\right] \end{aligned}$$

Next, applying the Poisson formula to h on $\mathbb{D}$ gives

$$f(z) = f(x+iy) = h(w) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{h}(\xi)\, \frac{1-|w|^2}{|\xi - w|^2}\, |d\xi|$$

With the Cayley transform $\xi = \frac{t-i}{t+i}$ and $w = \frac{z-i}{z+i}$, we obtain

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{h}\Big(\frac{t-i}{t+i}\Big)\, \frac{1-\left|\frac{z-i}{z+i}\right|^2}{\left|\frac{t-i}{t+i}-\frac{z-i}{z+i}\right|^2}\,\frac{2}{1+t^2}\,dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{f}(t)\,\frac{2y}{|t-z|^2}\,dt = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\,\tilde{f}(t)\,dt, \qquad y>0.
\end{aligned}
$$

□

Corollary 8.10. (*Boundary uniqueness theorem*) *If* $f \in H^p(\mathbb{C}_+)$, $1 \le p \le \infty$ *and* $f \neq 0$, *then*

$$\int_{\mathbb{R}} \frac{|\log|f(x)||}{1+x^2}dx < \infty.$$

Proof. Let $f \in H^p(\mathbb{C}_+) \implies f(z) = h(w)$ for $z \in \mathbb{C}_+, w \in \mathbb{D}$ and $h \in H^p(\mathbb{D})$ By the boundary uniqueness theorem for the disk:

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |\log|h(\xi)|||d\xi| < \infty \\
&\implies \frac{1}{2\pi}\int_{\mathbb{R}} \left|\log|\tilde{f}(t)|\right| \frac{2dt}{1+t^2} < \infty \\
&\implies \int_{\mathbb{R}} \frac{|\log|\tilde{f}(t)||}{1+t^2}dt < \infty.
\end{aligned}
$$

□

Corollary 8.11. (*Blaschke condition and Blaschke product*) *If* $f \in H^p(\mathbb{C}_+)$, $1 \le p \le \infty$, *and* $f \neq 0$, *then*

$$\sum \frac{\operatorname{Im}\lambda_n}{1+|\lambda_n|^2} < \infty,$$

where λ_n *are the zero of* f *in* $\mathbb{C}_+$ *(counting multiplicities). The corresponding Blaschke product (having similar properties as in* $\mathbb{D}$*) is*

$$B(z) = \prod_n \epsilon_n \frac{z-\lambda_n}{z-\bar{\lambda}_n},\ z \in \mathbb{C}_+,$$

where $\epsilon_n = \frac{|\lambda_n^2+1|}{\lambda_n^2+1}$ *(by definition,* $\epsilon_n = 1$ *for* $\lambda_n = i$*).*

Proof. Let $f \in H^P(\mathbb{C}_+), 1 \le p \le \infty$ and $f \neq 0$. Then there exists $g \in H^p(\mathbb{D})$ such that $U_p g = f$. Next, $f(\lambda_n) = 0 \implies U_p g(\lambda_n) = 0 \implies \left(\frac{1}{\pi(\lambda_n+i)^2}\right)^{1/p} g(\frac{\lambda_n - i}{\lambda_n+i}) = 0 \implies g(\gamma_n) =$

0 where $\gamma_n = \frac{\lambda_n - i}{\lambda_n + i} \in \mathbb{D}$. So $\lambda_n' s$ are roots of f if and only if $\gamma_n' s$ are roots of g.

$$\begin{aligned} |\gamma_n| &= \frac{|\lambda_n - i|}{|\lambda_n + i|}, \\ |\gamma_n|^2 = \gamma_n \overline{\gamma_n} &= \frac{\lambda_n - i}{\lambda_n + i} \cdot \frac{\overline{\lambda_n} + i}{\overline{\lambda_n} - i} \\ &= \frac{|\lambda_n|^2 + i(\lambda_n - \overline{\lambda_n}) + 1}{|\lambda_n|^2 - i(\lambda_n - \overline{\lambda_n}) + 1} = \frac{1 + |\lambda_n|^2 - 2y_n}{1 + |\lambda_n|^2 + 2y_n}. \end{aligned}$$

where $y_n = \text{Im}(\lambda_n)$.

Calculate $1 - |\gamma_n|^2 = \frac{4y_n}{1+|\lambda_n|^2+2y_n}$.

Since $g \in H^p(\mathbb{D})$, its zeros $\{\gamma_n\}$ satisfy the Blaschke condition

$$\sum_n (1 - |\gamma_n|) < \infty,$$

equivalently $\sum_n (1 - |\gamma_n|^2) < \infty$. Using the identity

$$1 - |\gamma_n|^2 = \frac{4y_n}{1 + |\lambda_n|^2 + 2y_n},$$

we see that $1 - |\gamma_n|^2$ is comparable to $\dfrac{y_n}{1 + |\lambda_n|^2}$ (indeed, the ratio tends to 4 as $|\lambda_n| \to 1$, hence $y_n \to 0$). Therefore

$$\sum_n (1 - |\gamma_n|^2) < \infty \quad \Longleftrightarrow \quad \sum_n \frac{y_n}{1 + |\lambda_n|^2} < \infty,$$

which is the desired Blaschke condition.

$$\frac{\dfrac{1 - |\gamma_n|^2}{\dfrac{y_n}{1 + |\lambda_n|^2}}}{} = \frac{\dfrac{4y_n}{1 + |\lambda_n|^2 + 2y_n}}{\dfrac{y_n}{1 + |\lambda_n|^2}} = \frac{4(1 + |\lambda_n|^2)}{1 + |\lambda_n|^2 + 2y_n} \longrightarrow 4,$$

since $|\lambda_n| \to 1$ forces $y_n = \Im \lambda_n \to 0$.

Hence by Comparison Test $\sum(1 - |\gamma_n|^2) < \infty \Leftrightarrow \sum \frac{y_n}{1+|\lambda_n|^2} < \infty$. Hence the desired Blaschke condition is: $\sum \frac{\text{Im}(\lambda_n)}{1+|\lambda_n|^2} < \infty$.

Blaschke factor computation. The Blaschke product for $g \in H^p(\mathbb{D})$ is

$$B_g(w) = \prod_n b_{\gamma_n} \frac{\gamma_n - w}{1 - \overline{\gamma_n} w}, \qquad w \in \mathbb{D},$$

where $b_{\gamma_n} = |\gamma_n|/\gamma_n$ (and $b_0 = 1$). Using $\gamma_n = \frac{\lambda_n - i}{\lambda_n + i}$ and $w = \frac{z-i}{z+i}$, one checks that

$$b_{\gamma_n} \frac{\gamma_n - w}{1 - \overline{\gamma_n} w} = \epsilon_n \frac{z - \lambda_n}{z - \overline{\lambda_n}}, \qquad \epsilon_n := \frac{|\lambda_n^2 + 1|}{\lambda_n^2 + 1}.$$

Therefore $B(z) := B_g\left(\frac{z-i}{z+i}\right)$ has the asserted form. If $\lambda_n = i$, then $\gamma_n = 0$ and the corresponding factor reduces to $w = \frac{z-i}{z+i}$, and indeed $\epsilon_n = 1$. □

Theorem 8.12. *Each function $f \in H^p(\mathbb{C}_+)$; $1 \le p \le \infty$, has a unique factorization of the form $f = \lambda BV[f]$, where $\lambda \in \mathbb{T}$, B is the Blaschke product constructed from the zeros of f, V is a singular inner function (an H^∞ function having no zeros in $\mathbb{C}_+$ and with unimodular boundary values on $\mathbb{R}$) of the form*

$$V(z) = e^{iaz} V_v(z) = e^{iaz} \exp\left(i \int_{\mathbb{R}} \frac{1 + tz}{t - z} dv(t) \right),$$

where $a \ge 0$, and v is a finite positive singular measure on $\mathbb{R}$, $[f]$ is the Schwarz-Herglotz outer factor of the form

$$[f](z) = \exp\left(\frac{1}{\pi i} \int_{\mathbb{R}} \frac{1 + tz}{t - z} \log |f(t)| \frac{dt}{1 + t^2} \right), \ z \in \mathbb{C}_+$$

Proof. Let $f \in H^p(\mathbb{C}_+)$. Then there exists $g \in H^p(\mathbb{D})$ such that $f(z) = g(w)$ for $z \in \mathbb{C}_+$ and $w \in \mathbb{D}$. Next

$$[g](w) = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{\xi + w}{\xi - w} \log |\tilde{g}(\xi)| |d\xi| \right]$$

Putting $\xi = \frac{t-i}{t+i}$ and $w = \frac{z-i}{z+i}$ we have:

$$\frac{t - i}{t + i} \pm \frac{z - i}{z + i} = \frac{\{tz + 1 + it - iz\} \pm \{tz + 1 - i(t - z)\}}{(t + i)(z + i)}$$

$$\implies \frac{\xi + w}{\xi - w} = \frac{1 + tz}{i(t - z)}$$

Hence $[f](z) = [g](\frac{z-i}{z+i}) = \exp\left(\frac{1}{\pi i} \int_{\mathbb{R}} \frac{1+tz}{t-z} \log |\tilde{f}(t)| \frac{dt}{1+t^2} \right), \ z \in \mathbb{C}_+$

Since $g \in H^p(\mathbb{D})$, it has a canonical factorization $g = \lambda BS[g]$. Here

$$S(w) = \exp\left[-\int_{\mathbb{T}} \frac{\xi + w}{\xi - w} \, d\mu(\xi) \right], \qquad w \in \mathbb{D},$$

for some finite positive measure $\mu \perp m$ on $\mathbb{T}$. Passing to the half-plane by $w = (z-i)/(z+i)$

and separating a possible atom at $1 \in \mathbb{T}$, we obtain

$$S(w) = \exp\left[-\frac{1+w}{1-w}\mu(\{1\}) - \int_{\mathbb{T}\setminus\{1\}} \frac{\xi+w}{\xi-w}\, d\mu(\xi)\right],$$
$$S\Big(\frac{z-i}{z+i}\Big) = \exp\left[i\mu(\{1\})z - \int_{\mathbb{R}} \frac{1+tz}{i(t-z)}\, d\mu\Big(\frac{t-i}{t+i}\Big)\right],$$

because $i(1+w)/(1-w) = -z$. Therefore the singular factor on $\mathbb{C}_+$ has the form

$$V(z) = e^{i\alpha z} \exp\left[\int_{\mathbb{R}} \frac{1+tz}{i(t-z)}\, d\nu(t)\right],$$

where $\alpha = \mu(\{1\})$ and ν is the pushforward of $\mu|_{\mathbb{T}\setminus\{1\}}$ under the Cayley transform. □

Remark 8.13 (Recovering atoms of the singular measure in $\mathbb{C}_+$). The canonical factorization on the half-plane has the same conceptual structure as in the disk setting: every nonzero $f \in H^p(\mathbb{C}_+)$ admits a factorization

$$f(z) = e^{iaz}\, B(z)\, S_\mu(z)\, O(z), \qquad z \in \mathbb{C}_+,$$

where $a \geq 0$, B is the (half-plane) Blaschke product of the zeros of f in $\mathbb{C}_+$, S_μ is the singular inner factor associated with a finite positive singular measure μ on $\mathbb{R}$, and O is an outer function determined by the boundary modulus of f (see Sections 6.2 and 6.3).

Two features are worth recording explicitly.

(i) *Local analyticity across the boundary.* If $f \in H^p(\mathbb{C}_+)$ extends analytically across an open interval $I \subset \mathbb{R}$, then the singular measure vanishes there: $\mu(I) = 0$. Equivalently, the boundary singularities of f are supported on $\operatorname{supp}\mu \subset \mathbb{R}$.

(ii) *Logarithmic residues (atoms on $\mathbb{R}$).* For $x_0 \in \mathbb{R}$, the atomic mass $\mu(\{x_0\})$ can be recovered from the boundary decay rate of $|f|$:

$$\mu(\{x_0\}) = -\pi \lim_{y\downarrow 0} y\, \log|f(x_0+iy)|,$$

whenever the limit exists (it always exists for $f = S_\mu$ and, more generally, it isolates the atomic part of μ). This is the half-plane analogue of the "logarithmic residue" computation on the disk, and follows from the identity

$$\log|S_\mu(x+iy)| = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}\, d\mu(t),$$

together with the fact that the Poisson kernel detects atoms.

Finally, the parameter a (the mass at ∞) is obtained from vertical growth:

$$a = -\lim_{y\to\infty} \frac{1}{y} \log |f(iy)|,$$

since $|e^{iaz}| = e^{-a\,\Im z}$ and the remaining factors contribute only sublinear growth along iy.

8.5 Invariant subspaces

Here we consider translation invariant subspaces of $L^2(\mathbb{R})$ and their Fourier dual objects - character invariant subspaces.

8.6 Duality between translation and multiplication by characters

Define the translation operator τ_s by

$$(\tau_s f)(x) = f(x - s),\ x \in \mathbb{R},\ \text{for } s \in \mathbb{R}.$$

This is a group of unitary operators on $L^2(\mathbb{R})$. A subspace $E \subset L^2(\mathbb{R})$ (closed, as always) is said to be (translation) 2-invariant and if $\tau_s E \subset E$ for all $s \in \mathbb{R}$, and (translation) 1-invariant if $\tau_s E \subset E$ for all $s \geq 0$ but not for (all) $s < 0$. The Fourier image of the translation operator τ_s is the multiplication operator by the corresponding character e^{isx} of the group $\mathbb{R}$:

$$\tau_s(\mathcal{F}f) = \mathcal{F}(e^{is}f),\ \text{for all } f \in L^2(\mathbb{R}).$$

Without any risk of confusion, we write e^{isx} both for the function $x \longmapsto e^{isx}$ and for the multiplication operator by this function, $f \longmapsto e^{isx}f$. Hence, we have

$$\tau_s = \mathcal{F} e^{isx} \mathcal{F}^{-1},$$

that is, the groups $(\tau_s)_{s\in\mathbb{R}}$ and $(e^{isx})_{s\in\mathbb{R}}$ are unitarily equivalent (conjugate) via the Fourier transform.

We use the same terminology as above for e^{isx} -invariant subspaces. A subspace $E \subset L^2(\mathbb{R})$ is (character) 2-invariant if $e^{isx}E \subset E$ for all $s \in \mathbb{R}$, and (character) 1-invariant if $e^{isx}E \subset E$ for $s \geq 0$ but for (all) $s < 0$. Hence, E is an 1- or 2- character invariant if and only if its Fourier image $\mathcal{F}E$ is a 1- or 2- translation invariant subspace.

Consequently, the Hardy space $H^2(\mathbb{C}_+)$ is a character 1-invariant subspace, and $\mathcal{F}H^2(\mathbb{C}_+) = L^2(\mathbb{R}_+)$ is translation 1-invariant.

Below, we will derive analogue of the Wiener theorem 3.5 and Beurling Helson theorem 3.6 for character invariant subspaces. First, we prepare the transfer of these results to $L^2(\mathbb{R})$ by means of the operator U_2.

Lemma 8.14. *Let* $u_s = \exp\left(s\frac{z+1}{z-1}\right)$ $s \in \mathbb{R}$, *and let* E *be a (closed) subspace of* $L^2(\mathbb{R})$. *The* E *is a 2-invariant subspace (with respect to the shift operator* $f \longmapsto zf$*) if and only if* $u_sE \subset E$, *for all* $s \in \mathbb{R}$, *and* E *is 1-invariant subspace if and only if* $u_sE \subset E$, *for all* $s \geq 0$, *but not for (all)* $s < 0$.

Proof. If $b \in H^\infty$, and E is a z-invariant subspace of $L^2(\mathbb{T})$, then $bE \subset E$. Indeed, by DCT, we have

$$\lim_{r\to 1} \|bf - b_rf\|_2 = 0, \text{ for all } f \in E,$$

where $b_r(z) = b(rz)$.

On the other hand, $z^nf \in E$ for every $n \geq 0$, and therefore $b_rf \in E$, since the Taylor series of b_r converges absolutely on $\mathbb{T}$. Hence $bf \in E$. The same argument applies to $\overline{b} \in H^\infty$ and $\overline{z}$-invariant subspaces E. This proves the "only if" part of the lemma.

For the converse, it suffices to show that the function z is the bounded pointwise limit of the functions

$$\phi_s = \frac{u_s - (1-s)}{u_s - (1+s)} \qquad (s \to 0_+).$$

We have $\mathrm{Re}(1 - u_s(\zeta)) \geq 0$, and hence $|\phi_s(\zeta)| \leq 1$ for $\zeta \in \mathbb{T}$. On the other hand, using the standard expansion $e^{sw} = 1 + sw + o(s)$ as $s \to 0_+$, we obtain

$$\lim_{s\to 0} \phi_s(\zeta) = \zeta, \qquad \zeta \in \mathbb{T} \setminus \{1\}.$$

□

Theorem 8.15. (*P. Lax, 1959*) *Let* E *be a subspace of* $L^2(\mathbb{R})$.

(i) E *is a (character) 2-invariant subspace if and only if* $E = \chi_\Sigma L^2(\mathbb{R})$ *for a measurable subset* $\Sigma \subset \mathbb{R}$.

(ii) E *is a (character) 1-invariant subspace if and only if* $E = \mathcal{F}_q H^2(\mathbb{C}_+)$ *for a measurable function* q *on* $\mathbb{R}$ *with* $|q| = 1$ *a.e.*

Proof. Lemma 8.14 shows that E is 2 or 1-invariant if and only if its preimage $U_2^{-1}E \subset L^2(\mathbb{T})$ has the same property with respect to the shift operator on $L^2(\mathbb{R})$. The results thus follow by applying theorems 3.5, 3.6 and Theorem 8.7. $\square$

Corollary 8.16. *Let E be a subspace of $L^2(\mathbb{R})$.*

1. *E is translation 2-invariant if and only if $E = \mathcal{F}\chi_\Sigma L^2(\mathbb{R})$ for a measurable subset $\Sigma \subset \mathbb{R}$.*
2. *E is translation 1-invariant if and only if $E = \mathcal{F}qH^2(\mathbb{C}_+)$ for a measurable function q on $\mathbb{R}$ with $|q| = 1$ a.e.*

Indeed, it suffices to use Theorem 8.15 and duality of Subsection 8.6.

Corollary 8.17. *(i) If $F \subset H^2(\mathbb{C}_+)$, then $\overline{\text{span}}_{H^2_+}\{e^{isx}F : s \geq 0\} = \theta H^2(\mathbb{C}_+)$, where θ is the g.c.d of the inner factors of $f \in F$.*

(ii) If $F \subset L^2(\mathbb{R}_+)$, then $\overline{\text{span}}_{L^2(\mathbb{R}_+)}\{\tau_s F : s \geq 0\} = \mathcal{F}(\theta H^2(\mathbb{C}_+))$, where θ is the g.c.d of the inner factors of $\mathcal{F}^{-1}f$, $f \in F$.

(iii) If $f \in L^2(\mathbb{R})$, then $\overline{\text{span}}_{L^2(\mathbb{R})}\{e^{isx}f : s \in \mathbb{R}\} = L^2(\mathbb{R})$ if and only if $f \neq 0$ a.e. on $\mathbb{R}$.

(iv) If $f \in L^2(\mathbb{R})$, then $\overline{\text{span}}_{L^2(\mathbb{R})}\{e^{isx}f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $f \neq 0$ a.e. and

$$\int_{\mathbb{R}} (1 + x^2) \log |f| dx = -\infty$$

(v) If $f \in L^2(\mathbb{R})$, then $\overline{\text{span}}_{L^2(\mathbb{R})}\{\tau_s f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $\mathcal{F}f \neq 0$ a.e. on $\mathbb{R}$

(vi) If $f \in L^2(\mathbb{R})$, then $\overline{\text{span}}_{L^2(\mathbb{R})}\{\tau_s f : s \geq 0\} = L^2(\mathbb{R})$ if and only if $\mathcal{F}f \neq 0$ a.e. and

$$\int_{\mathbb{R}} (1 + x^2) \log |\mathcal{F}f| dx = -\infty.$$

Indeed, it suffices to use Theorem 8.15 and Corollary 8.16 and the corresponding properties of z-invariant subspaces of $L^2(\mathbb{R})$.

Theorem 8.18 (Cauchy representation)**.** *Assume that $1 \leq p < \infty$.*

(i) Let $F \in H^p(\mathbb{C}_+)$, and let $F(x)$ denote its boundary function. Then $F(x) \in L^p(-\infty, \infty)$ and

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t - z}\, dt, \qquad y > 0, \tag{8.6.1}$$

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t - z}\, dt, \qquad y < 0. \tag{8.6.2}$$

(ii) *Let $F(x) \in L^p(-\infty,\infty)$ satisfy* (8.6.1). *Then the Cauchy formula above and the Poisson representation from Corollary 8.9 define the same function on $\mathbb{C}_+$. This function belongs to $H^p(\mathbb{C}_+)$, and its non-tangential boundary function is equal to $F(x)$ a.e.*

Proof. (i) By Fatou's lemma and the definition of $H^p(\mathbb{C}_+)$, we have:

$$\int_{-\infty}^{\infty} |F(x)|^p\, dx \le \liminf_{y\to 0} \int_{-\infty}^{\infty} |F(x+iy)|^p dx < \infty \implies F \in L^p(-\infty,\infty).$$

Let $G(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t)}{t-z}dt, y \neq 0$ Then $G(z)$ is holomorphic separately for $y > 0$ and $y < 0$. For $y > 0$

$$\begin{aligned} G(z) - G(\overline{z}) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{1}{t-z} - \frac{1}{t-\overline{z}}\right]F(t)dt \\ &= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{F(t)}{(t-x)^2+y^2}dt \\ &= F(z). \end{aligned}$$

Since $F(z)$ and $G(z)$ are holomorphic on $\mathbb{C}_+$ so is $G(\overline{z}), z \in \mathbb{C}_+$. But

$$\overline{G(\overline{z})} = -\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\overline{F}(t)}{t-z}dt, z \in \mathbb{C}_+$$

is also holomorphic. Since $\overline{G}(\overline{z})$ and $G(\overline{z})$ are both holomorphic, hence $G(\overline{z})$ is constant on $\mathbb{C}_+$. Since $G(-iy) \to 0$ as $y \to \infty$, $G(\overline{z}) \equiv 0$ on $\mathbb{C}_+$. Thus (8.6.1) holds.

(ii) Assuming

$$0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(t)}{t-z}dt, \forall y < 0 \implies 0 = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{F(t)}{t-\overline{z}}dt, \forall y > 0 \implies 0 = G(\overline{z}), \forall y > 0$$

In (i) we have proved: $G(z) - G(\overline{z}) = F(z)$ for $y > 0 \implies G(z) = F(z)$. Applying

Hölder's inequality:

$$\begin{aligned}
\int_{-\infty}^{\infty} |F(x+iy)|^p dx &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{y/\pi}{(t-x)^2+y^2} F(t) dt \right|^p dx \\
&= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \left[\frac{y/\pi}{(t-x)^2+y^2} F(t) \right]^{1/p} \left[\frac{y/\pi}{(t-x)^2+y^2} \right]^{1/q} dt \right|^p dx \\
&\leq \int_{-\infty}^{\infty} \left| \left(\int_{-\infty}^{\infty} \frac{y/\pi}{(t-x)^2+y^2} |F(t)| dt \right)^{1/p} \left(\int_{-\infty}^{\infty} \frac{y/\pi}{(t-x)^2+y^2} dt \right)^{1/q} \right|^p dx \\
&\leq \int_{-\infty}^{\infty} \left| \left(\int_{-\infty}^{\infty} \frac{y/\pi}{(t-x)^2+y^2} |F(t)| dt \right) \left(\int_{-\infty}^{\infty} \frac{y/\pi}{(t-x)^2+y^2} dt \right)^{p/q} \right| dx \\
&\leq \int_{-\infty}^{\infty} \left| \left(\int_{-\infty}^{\infty} \frac{y/\pi}{(t-x)^2+y^2} |F(t)|^p dt \right) \right| dx \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{y/\pi}{(t-x)^2+y^2} |F(t)|^p dx dt \\
&\leq \int_{-\infty}^{\infty} |F(t)|^p dt
\end{aligned}$$

This shows that $F \in H^p(\mathbb{C}_+)$ □

8.7 Cauchy kernels and L^p-decomposition

Theorem 8.19. *(i) Show that* $H^p(\mathbb{C}_+) = \overline{\text{span}}_{L^2(\mathbb{R})} \left\{ \frac{1}{x-\bar{\mu}} : \text{Im}\,\mu > 0 \right\}$ *for* $1 \leq p \leq \infty$. *(Hint: Use* $H^p(\mathbb{C}_+) = U_p H^p$ *and solve* $U_p f = \frac{1}{x-\bar{\mu}}$*).*

(ii) Let $1 < p < \infty$. *Show that* $L^p(\mathbb{R}) = H^p(\mathbb{C}_+) \oplus H^p(\mathbb{C}_-)$, *where* $\oplus$ *stands for the orthogonal sum for* $p = 2$ *and direct sum for* $p \neq 2$.

(iii) Let

$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z} dt, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

be the Cauchy integral of $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, *then the following are equivalent.*

(a) $f \in H^p(\mathbb{C}_+)$.

(b) $Cf = f_*$, *where* f_* *stands for the Poisson integral extension.*

(c) $Cf(z) = 0$ *for* $\text{Im}\, z < 0$.

Proof. Previously solved. □

Theorem 8.20. (*The Paley Wiener theorem*) *An entire function E is called of exponential type if*

$$\overline{\lim}_{|z|\to\infty}\frac{\log|E(z)|}{|z|}<\infty;$$

the limit itself is the type *of E. Let $\mathcal{E}_a$ = set of all entire functions of exponential type $\le a$. For $a>0$, show that the following are equivalent.*

(i) $E\in\mathcal{E}_a$ and $E|_{\mathbb{R}}\in L^2(\mathbb{R})$.

(ii) There exists $f\in L^2(\mathbb{R})$ such that $\mathcal{F}f=E$ and supp $f\in[-a,a]$.

Hint: For (ii) $\implies$ (i), estimate the exponential type of E applying the Cauchy inequality to the Fourier transform of f :

$$|E(z)|=\Big|\int_{-a}^{a}e^{-ixz}f(x)dx\Big|\le\|f\|_2\Big(\frac{e^{2a|\operatorname{Im}z|}-1}{\operatorname{Im}z}\Big)^{\frac12}\le(2a)^{\frac12}e^{a|\operatorname{Im}z|}.$$

Moreover, $\|E\|_2=\|f\|_2$ by Plancherel's theorem:

(i) $\implies$ (ii): First suppose that $E|_{\mathbb{R}}\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. Then by Phragmèn-Lindelöf theorem $|E(z)|\le\|E\|_\infty e^{a|\operatorname{Im}z|}$, for $z\in\mathbb{C}$, implies

$$|E_\lambda(z)|=\frac{i\lambda}{z+i\lambda}e^{aiz}E(z)\in H^2(\mathbb{C}_+),\ \ \lambda>0.$$

The Paley Wiener theorem 8.8 entails that $\mathcal{F}(E_\lambda)=0$ a.e. on $(-\infty,0)$ and hence $\mathcal{F}(e^{aiz}E)=0$ on $(-\infty,a)$ (because $\lim\limits_{\lambda\to\infty}\|E_\lambda-e^{aiz}E\|_{L^2(\mathbb{R})}=0$). Therefore, $\mathcal{F}(E)=\tau_a\mathcal{F}(e^{iaz}E)=0$ a.e on $(-\infty,-a)$. Similarly $\mathcal{F}(E)=0$ a.e. on $(a,\infty.)$ and we get (ii).

In general case, replace E by $E^\epsilon(z)=\int_{\mathbb{R}}E(z-t)\phi_\epsilon(t)dt$, where $\phi_\epsilon(t)=\epsilon^{-1}\phi(\frac{t}{\epsilon})$, $\phi\ge0$ is compactly supported in $\mathbb{R}$. It follows that $E^\epsilon\in\mathcal{E}_{a+\epsilon}$ and supp $(E^\epsilon)\subset[-a-\epsilon,a+\epsilon]$, and we have $\lim\limits_{\epsilon\to0}\|E^\epsilon-E\|_{L^2(\mathbb{R})}=0$.

Question 8.21. (a) Show that $f\in H^2(\mathbb{C}_+)$ if and only if $f\in L^2(\mathbb{R})$ and $\mathcal{F}(f)=0$ a.e. on $\mathbb{R}$.

(b) Find $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ such that $L^2(\mathbb{R})=\overline{\operatorname{span}}_{L^2(\mathbb{R})}(\tau_s f:s\in\mathbb{R})$ and $L^1(\mathbb{R})\ne\overline{\operatorname{span}}_{L^1(\mathbb{R})}\{\tau_s f:s\in\mathbb{R}\}$ (Hint: Consider $f=\chi_{(a,b)}$.)

(c) **F. & M. Riesz theorem for** $\mathbb{R}$: Let μ be a complex Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}e^{ist}d\mu(t)=0$ for all $s>0$. Show that $\mu<<m$.

8.8 Exercises and extensions

This final local exercise block is intended as a bridge from the disk theory to the upper half-plane model.

Core exercise

Exercise 8.22. $H^1(\mathbb{C}_+) = H^2(\mathbb{C}_+)H^2(\mathbb{C}_+)$

Proof. We know that $w : \mathbb{D} \to \mathbb{C}_+$ is a conformal map.

$$\begin{aligned} F \in H^1(\mathbb{C}_+) &\implies F \cdot w' \in H^1(\mathbb{D}) \\ &\implies F \cdot w' = G_1 \cdot G_2 \text{ where } G_1, G_2 \in H^2(\mathbb{D}) \\ &\implies F = [G_1 \cdot (w')^{-1/2}][G_2 \cdot (w')^{-1/2}] \end{aligned}$$

Next define two functions g_1, g_2 by the following forms:

$$\begin{aligned} &g_1 \bullet w = G_1 \cdot (w')^{-1/2} \\ &g_2 \bullet w = G_2 \cdot (w')^{-1/2} \\ &\implies (g_1 \bullet w)(w')^{1/2} = G_1 \in H^2(\mathbb{D}) \\ &\implies (g_2 \bullet w)(w')^{1/2} = G_2 \in H^2(\mathbb{D}) \\ &\implies g_1, g_2 \in H^2(\mathbb{C}_+) \end{aligned}$$

and

$$\begin{aligned} &F = (g_1 \bullet w)(g_2 \bullet w) \\ &\implies F \bullet w^{-1} = g_1 \cdot g_2 \\ &\implies f = g_1 \cdot g_2. \end{aligned}$$

□

Chapter 9

Problem Sets and Further Reading

This chapter is a self-contained bank of problem sets aligned one-to-one *with the syllabus items and the chapter sequence of these notes. Each set is organized from foundational checks to more conceptual questions, and can be used for tutorials, take-home assignments, or midterm/final preparation.*

Learning objectives.

- Revisit each chapter through carefully staged problems rather than isolated drill.
- Separate routine verification from deeper synthesis and open-ended extensions.
- Provide a clean assignment architecture suitable for tutorials, qualifying-exam preparation, and independent reading courses.

Problem-set architecture.

- In each set, the first problems are intended as *warm-up* checks of definitions and basic lemmas.
- The middle problems form the *core track*; they assemble the main ideas of the chapter into full proofs.
- The final problems are *challenge/extension* problems meant for seminar discussion, projects, or exam enrichment.

9.1 Problem Set I: Invariant subspaces of $L^2(\mathbb{T}, \mu)$

Problem-set architecture.

- Warm-up: Problems 1–4.
- Core track: Problems 5–9.
- Challenge track: Problems 10–13.

1. Let μ be a finite Borel measure on $\mathbb{T}$ and let M_z denote multiplication by z on $L^2(\mu)$. Show that a closed subspace $E \subset L^2(\mu)$ is *reducing* for M_z (i.e. invariant for both M_z and M_z^*) if and only if $E = \chi_\sigma L^2(\mu)$ for some Borel set $\sigma \subset \mathbb{T}$.

2. For $f \in L^2(\mu)$ define the cyclic subspace
$$E_f := \overline{\text{span}}\{z^n f : n \geq 0\} \subset L^2(\mu).$$
Show that E_f is invariant for M_z, and compute the *wandering subspace* $E_f \ominus zE_f$ in terms of the orthogonal projection of f onto $E_f \ominus zE_f$.

3. (Unilateral shift model) Let $\mu = m$ be normalized arc-length measure on $\mathbb{T}$. Show that $H^2 = \overline{\text{span}}\{z^n : n \geq 0\}$ is invariant but not reducing. Compute $H^2 \ominus zH^2$ and identify the shift $M_z|_{H^2}$ as a unilateral shift of multiplicity 1.

4. (Doubly invariant $\Rightarrow$ characteristic function) Let μ be a finite Borel measure on $\mathbb{T}$. Assume $E \subset L^2(\mu)$ satisfies $zE = E$. Show that E is reducing and hence $E = \chi_\sigma L^2(\mu)$ for some Borel σ. (*Hint:* use $z^{-1}E \subset E$ and the functional calculus for normal operators.)

5. (Simply invariant subspaces and uniqueness) Let $\mu = m$ and let $E \subset L^2(\mathbb{T}, m)$ be a nonzero closed subspace with $zE \subsetneq E$. Show that E contains a *wandering vector* $w \neq 0$ such that $\{z^n w : n \geq 0\}$ is orthogonal. Deduce that $E = \theta H^2$ for an inner function θ (Beurling's theorem).

6. (Boundary uniqueness) Let $f \in H^p(\mathbb{D})$ for some $0 < p \leq \infty$. Assume the non-tangential boundary values $\widetilde{f}$ vanish on a set $E \subset \mathbb{T}$ of positive Lebesgue measure. Prove that $f \equiv 0$ on $\mathbb{D}$. (*Hint:* reduce to the outer factor and use $\log |f|$ integrability.)

7. Let μ be a finite Borel measure on $\mathbb{T}$ with Lebesgue decomposition $\mu = \mu_a + \mu_s$ with respect to m. Show that

$$H_0^2(\mu) := zH^2(\mu) = H_0^2(\mu_a) \oplus L^2(\mu_s).$$

Conclude that $H^2(\mu)$ is a proper subspace of $L^2(\mu)$ if and only if μ has a nontrivial absolutely continuous part.

8. (Distance to $H_0^2(\mu)$) With $H_0^2(\mu)$ as above, show that for $f \in L^2(\mu)$,

$$\operatorname{dist}(f, H_0^2(\mu)) = \left|\langle f, 1\rangle_{L^2(\mu)}\right| \cdot \|P_{\mathbb{C}\cdot 1}^{\perp}\|^{-1},$$

and specialize this formula to $\mu = m$. (*Hint:* identify $(H_0^2(\mu))^{\perp}$ and use Riesz representation.)

9. Let μ be a finite Borel measure on $\mathbb{T}$. Prove that the following are equivalent:

   (i) There exists a non-reducing invariant subspace $E \subset L^2(\mu)$ with $zE \subset E$.

   (ii) There exists a nonzero complex measure $\nu \ll \mu$ such that $\int_{\mathbb{T}} z^n \, d\nu = 0$ for all $n \geq 1$.

10. Let $0 \leq \mu \ll m$ be a finite measure on $\mathbb{T}$. Can $H^2(\mu)$ be a proper *reducing* subspace of $L^2(\mu)$? Give a complete answer (prove or produce a counterexample).

11. Let $f \in H^2$ be *outer*. Show that $\overline{\operatorname{span}}\{z^n f : n \geq 0\} = H^2$, and that

$$\overline{\operatorname{span}}\{z^n f : n \geq 1\} = zH^2.$$

(*Interpretation:* cyclicity of outer functions.)

12. (Factorization in L^2) Let (X, μ) be a finite measure space. Decide whether it is true that every $F \in L^2(\mu)$ can be written as $F = GH$ with $G, H \in L^2(\mu)$. Give a sharp condition on (X, μ) under which such a factorization always exists.

13. (Wold decomposition viewpoint) Let V be an isometry on a Hilbert space $\mathcal{H}$ and let $\mathcal{W} := \mathcal{H} \ominus V\mathcal{H}$. Prove the Wold decomposition

$$\mathcal{H} = \Big(\bigoplus_{n\geq 0} V^n \mathcal{W}\Big) \oplus \mathcal{H}_u,$$

where V is unitary on $\mathcal{H}_u$. Apply this to $V = M_z|_E$ for an invariant subspace $E \subset L^2(\mathbb{T}, m)$ and interpret $\dim(E \ominus zE)$ as the shift multiplicity.

9.2 Problem Set II: Applications, outer functions, and weighted approximation

Problem-set architecture.

- Warm-up: Problems 1–3.
- Core track: Problems 4–8.
- Challenge track: Problems 9–11.

1. Let $f \in H^2$. Show that $\|f\|_{H^2} = \sup_{0<r<1} \|f_{(r)}\|_{L^2(\mathbb{T})}$ where $f_{(r)}(\zeta) := f(r\zeta)$.
2. Let p be a polynomial. Prove that p is outer (as an H^2-function) if and only if p has no zeros in $\mathbb{D}$. Deduce that $z - \lambda$ is outer if and only if $|\lambda| \geq 1$.
3. Let $w \in L^1_+(\mathbb{T}, m)$ and assume there exists $f \in H^2$ such that $|f|^2 = w$ a.e. on $\mathbb{T}$. Show that there exists a *unique* outer function $f_o \in H^2$ (unique up to a unimodular constant, fixed by $f_o(0) > 0$) such that $|f_o|^2 = w$ a.e.
4. (Szegő infimum) For $w \in L^1_+(\mathbb{T}, m)$ define
$$\delta(w) := \inf\Big\{ \int_{\mathbb{T}} |1 - p|^2 w \, dm : \; p \text{ is a polynomial with } p(0) = 0 \Big\}.$$
Assume $\log w \in L^1(\mathbb{T})$. Show that $\delta(w) = \exp\Big(\int_{\mathbb{T}} \log w \, dm \Big)$. (*Hint:* use outer functions and orthogonal projection onto $H^2_0(w\, dm)$.)
5. (Weighted polynomial approximation) Let $w \in L^1_+(\mathbb{T})$ and consider $L^2(w\, dm)$. Show that polynomials are dense in $L^2(w\, dm)$ if and only if $\delta(w) = 0$. Relate this to the condition $\log w \notin L^1$.
6. (Probabilistic/prediction interpretation) Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary process with spectral measure μ on $\mathbb{T}$. Formulate the one-step prediction problem and show that the optimal prediction error equals $\operatorname{dist}(1, H^2_0(\mu))$. Identify the Szegő condition in this language.

7. Let $f \in H^\infty \cap H^2$ with $\|f\|_\infty < 1$. Show that $1+f$ is outer. (*Hint:* use $\Re \log(1+f) = \log|1+f|$ and harmonic majorants.)

8. Let $f \in H^2 \cap L^\infty(\mathbb{T})$. Show that $e^f \in H^2$ and is outer.

9. (Arithmetic of inner functions) Let θ_1, θ_2 be inner functions. Define their greatest common inner divisor $\gcd(\theta_1, \theta_2)$ and least common inner multiple $\operatorname{lcm}(\theta_1, \theta_2)$. Prove that
$$\theta_1 H^2 + \theta_2 H^2 = \gcd(\theta_1, \theta_2)\, H^2, \qquad \theta_1 H^2 \cap \theta_2 H^2 = \operatorname{lcm}(\theta_1, \theta_2)\, H^2.$$

10. Let $f \in H^1 \cap L^\infty$. Show that there exist $g, h \in L^2(\mathbb{T})$ such that $f^2 = gh$ and each of g, h lies in the cyclic subspace generated by f. (*Aim:* practice L^2-factorization and invariant subspaces.)

11. (Concrete Szegő computations) Compute $\delta(w)$ explicitly for the following weights:

    (a) $w(\zeta) = |1 - a\zeta|^2$ with $|a| < 1$;

    (b) $w(\zeta) = \exp(2\Re(b\zeta))$ with $b \in \mathbb{C}$;

    (c) $w(\zeta) = |\zeta - \lambda|^\alpha$ for $|\lambda| = 1$ and $\alpha > -1$.

    State precisely for which parameters $\log w \in L^1$ and interpret the result.

9.3 Problem Set III: H^p on the disk, boundary limits, and canonical factorization

Problem-set architecture.

- Warm-up: Problems 1–4.
- Core track: Problems 5–8.
- Challenge track: Problems 9–11.

1. Prove that $H^p(\mathbb{D}) \subset H^q(\mathbb{D})$ whenever $0 < q < p \le \infty$ and give an example showing the inclusion is strict.

2. Let $f \in \operatorname{Hol}(\mathbb{D})$ and define $u_r(\zeta) = |f(r\zeta)|^p$ for $0 < r < 1$. Show that u_r increases in r if $f \in H^p$ and explain how this yields the existence of radial boundary values in L^p for $p \ge 1$.

3. (Hardy–Littlewood maximal theorem for H^p) Let $0 < p < \infty$ and let $f \in H^p$. Define the non-tangential maximal function

$$f^*(\xi) := \sup_{z \in \Gamma(\xi)} |f(z)|.$$

Show that $f^* \in L^p(\mathbb{T})$ and that $\|f^*\|_p \lesssim \|f\|_{H^p}$. Deduce Fatou's theorem for non-tangential limits.

4. (Jensen formula and inequality) Let $f \in \mathrm{Hol}(\mathbb{D})$, $f(0) \neq 0$, and let (a_n) be the zeros of f in $\mathbb{D}$ counted with multiplicity. Prove Jensen's formula and deduce Jensen's inequality

$$\log |f(0)| \leq \int_{\mathbb{T}} \log |\widetilde{f}| \, dm.$$

5. (Blaschke condition) Let $(a_n) \subset \mathbb{D}$ be a sequence without accumulation in $\mathbb{D}$. Show that the Blaschke product

$$B(z) = \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

converges uniformly on compact subsets of $\mathbb{D}$ if and only if $\sum_n (1 - |a_n|) < \infty$.

6. (Singular inner functions) Let σ be a finite positive singular measure on $\mathbb{T}$. Show that

$$S_\sigma(z) = \exp\left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\sigma(\zeta) \right)$$

defines an inner function, identify its boundary modulus, and prove that S_σ has no zeros in $\mathbb{D}$.

7. (Riesz–Smirnov factorization) Let $0 < p \leq \infty$ and $f \in H^p$, $f \not\equiv 0$. Prove that there exist an inner function θ and an outer function F such that $f = \theta F$. If $p < \infty$, refine this to the full canonical factorization $f = B \, S_\sigma \, F$.

8. (Outer criterion) Let F be outer. Show that $\log |F| \in L^1(\mathbb{T})$ and that

$$\log |F(0)| = \int_{\mathbb{T}} \log |\widetilde{F}| \, dm.$$

Conversely, show that if $f \in H^p$ and equality holds in Jensen's inequality, then f is outer up to a unimodular constant.

9. (Boundary regularity and $0 < p < 1$) Let $0 < p < 1$. Investigate which parts of Fatou's

theorem and canonical factorization remain valid in H^p. In particular, decide whether non-tangential boundary limits coincide with radial limits a.e. for H^p when $0 < p < 1$.

10. (Approximation by Blaschke products) Let θ be an inner function. Show that θ can be approximated uniformly on compact subsets of $\mathbb{D}$ by finite Blaschke products if and only if its singular inner part is trivial. State the precise approximation theorem you use.

11. (Zero sets and uniqueness) Let $E \subset \mathbb{T}$ have positive measure. Construct a nontrivial outer function F with boundary values $\widetilde{F} = 0$ on E and $\widetilde{F} \neq 0$ a.e. on $\mathbb{T} \setminus E$. Explain why this does not contradict boundary uniqueness for *analytic* functions with L^p boundary values.

9.4 Problem Set IV: Szegő infimum, Smirnov class, and Phragmén–Lindelöf

Problem-set architecture.

- Warm-up: Problems 1–3.
- Core track: Problems 4–6.
- Challenge track: Problems 7–9.

1. Show that the Smirnov class $N_+(\mathbb{D})$ is a vector space and that $H^p \subset N_+$ for every $0 < p \leq \infty$.

2. Let φ be a conformal automorphism of $\mathbb{D}$. Show that $f \mapsto f \circ \varphi$ preserves $N_+(\mathbb{D})$ and sends outer functions to outer functions (up to a unimodular constant).

3. (Locally outer) Let $f \in N_+(\mathbb{D})$. Show that for each $0 < r < 1$ there exists an outer function F_r on $\mathbb{D}$ such that f/F_r is bounded and analytic on $r\mathbb{D}$.

4. Prove the generalized Phragmén–Lindelöf principle in the form stated in the notes, and deduce the classical maximum modulus principle for H^∞ as a corollary.

5. (Outer on a smaller disk) Let $f \in N_+(\mathbb{D})$. Assume f is outer on $\frac{1}{2}\mathbb{D}$. Decide whether f must be outer on $\mathbb{D}$. Give a proof or counterexample, and state clearly which step fails if the implication is false.

6. (Szegő and N_+) Let $w \in L^1_+(\mathbb{T})$. Show that $\log w \in L^1$ if and only if there exists $F \in N_+$ such that $|\widetilde{F}|^2 = w$ a.e. Relate this to the existence of an outer H^2-function when additionally $w \in L^1$.

7. (Frostman shifts) Let θ be inner and $\alpha \in \mathbb{D}$. Show that

$$\theta_\alpha(z) := \frac{\theta(z) - \alpha}{1 - \overline{\alpha}\,\theta(z)}$$

is inner, and study how the singular measure in the canonical factorization changes under this transformation.

8. (A sharp integrability test) Let F be outer with boundary modulus $|\widetilde{F}|$. Give a necessary and sufficient condition (in terms of $\log|\widetilde{F}|$) for $1/F \in H^p$. Check your condition on a typical outer function with a power singularity at one boundary point.

9. (Szegő through Toeplitz forms) For $w \in L^1_+$ define the Toeplitz quadratic form on polynomials by

$$Q_w(p) := \int_{\mathbb{T}} |p|^2 w\, dm.$$

Show that the Szegő infimum equals the inverse of the squared norm of the evaluation functional at 0 on the closure of polynomials in $L^2(w\,dm)$. Use this to give an operator-theoretic proof of the Szegő formula under $\log w \in L^1$.

9.5 Problem Set V: Harmonic conjugates, Hilbert transform, and Helson–Szegő

Problem-set architecture.

- Warm-up: Problems 1–4.
- Core track: Problems 5–7.
- Challenge track: Problems 8–9.

1. Let $u \in L^1(\mathbb{T})$ with $\int_{\mathbb{T}} u\, dm = 0$. Show that the harmonic conjugate $\widetilde{u}$ is defined a.e. (via principal value) and satisfies $\int_{\mathbb{T}} \widetilde{u}\, dm = 0$.

2. Compute the Hilbert transform of $\cos(n\theta)$ and $\sin(n\theta)$ for $n \geq 1$ and deduce that the conjugation operator is an isometry on the subspace of real trigonometric polynomials with zero mean.

3. (M. Riesz theorem) Prove that the Hilbert transform extends to a bounded operator on $L^p(\mathbb{T})$ for $1 < p < \infty$, and explain why it fails on L^1 and L^∞.

4. Show that the Riesz projection $P_+ : L^2(\mathbb{T}) \to H^2$ extends boundedly to $L^p(\mathbb{T})$ for $1 < p < \infty$ and identify P_+ in terms of the Hilbert transform.

5. (Generalized Fourier series) Let μ be a finite measure on $\mathbb{T}$ and consider the system $\{z^n\}_{n\in\mathbb{Z}}$ in $L^2(\mu)$. Give necessary and sufficient conditions for $\{z^n\}_{n\in\mathbb{Z}}$ to be complete, and relate this to the support of μ and to invariant subspaces of $L^2(\mu)$.

6. (Helson–Szegő weights, conceptual form) Let $w \in L^1_+(\mathbb{T})$. Show that the following are equivalent:

   (i) The Riesz projection P_+ is bounded on $L^2(w\,dm)$.

   (ii) There exist bounded real functions $u, v \in L^\infty(\mathbb{T})$ with $\|v\|_\infty < \frac{\pi}{2}$ such that

$$w = e^{u+\widetilde{v}} \quad \text{a.e.}$$

   (*This is the Helson–Szegő theorem; your task is to prove the direction emphasized in the notes and outline the converse.*)

7. Show that if $w \in A_2$ (Muckenhoupt class) then the Hilbert transform is bounded on $L^2(w\,dm)$. Compare this sufficient condition with the Helson–Szegő characterization.

8. Let $f \in H^2$ be outer and write $f = e^g$ with $g \in H^1$. Show that $|f|^2 = e^{2\Re g}$ and interpret Helson–Szegő in terms of $\Im g$ (an L^∞ control on the argument). Give an example where $\Re g \in L^1$ but $\Im g \notin L^\infty$ and explain what fails.

9. (Bases of exponentials) Let μ be an absolutely continuous measure with density w. Investigate when $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in $L^2(w\,d\theta)$. State a sharp criterion in terms of w (or $\log w$) as far as covered in the notes, and illustrate it on explicit families of weights.

9.6 Problem Set VI: Transfer to the upper half-plane and $H^p(\mathbb{C}_+)$

Problem-set architecture.

- Warm-up: Problems 1–3.
- Core track: Problems 4–6.
- Challenge track: Problems 7–8.

1. Let $\phi(z) = \frac{z-i}{z+i}$ be the Cayley transform mapping $\mathbb{C}_+$ conformally onto $\mathbb{D}$. Show that ϕ extends to a.e. boundary map $\mathbb{R} \to \mathbb{T}$ and compute the Jacobian relating $d\theta$ on $\mathbb{T}$ to dx on $\mathbb{R}$.

2. Define $H^p(\mathbb{C}_+)$ via non-tangential maximal functions (or via boundary values on $\mathbb{R}$). Show that $f \mapsto (f \circ \phi^{-1}) \cdot (\phi^{-1})'^{1/p}$ gives an isomorphism between $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{D})$ (with the usual convention for $p = \infty$).

3. (Poisson and Cauchy kernels) Derive the Poisson kernel $P_y(x-t)$ for $\mathbb{C}_+$ and show that for $F \in L^p(\mathbb{R})$, the Poisson extension

$$u(x+iy) = (P_y * F)(x)$$

is harmonic in $\mathbb{C}_+$ and has non-tangential boundary values F a.e.

4. (Paley–Wiener theorem for $H^2(\mathbb{C}_+)$) Show that $f \in H^2(\mathbb{C}_+)$ if and only if there exists $g \in L^2(0,\infty)$ such that

$$f(z) = \int_0^\infty e^{izt}\, g(t)\, dt, \qquad z \in \mathbb{C}_+,$$

and that $\|f\|_{H^2} = \|g\|_{L^2(0,\infty)}$.

5. (Inner–outer factorization on $\mathbb{C}_+$) State and prove the canonical factorization in $H^p(\mathbb{C}_+)$. In particular, identify:

   (a) Blaschke products in $\mathbb{C}_+$ and their convergence condition;

   (b) singular inner factors coming from singular measures on $\mathbb{R}$;

(c) the outer factor written via $\log|\widetilde{f}|$ on $\mathbb{R}$.

6. (Invariant subspaces) Formulate the analogue of Beurling's theorem for the shift-invariant subspaces of $H^2(\mathbb{C}_+)$ (multiplication by $e^{it\cdot}$ or by the Cayley image of z). Explain how inner functions on $\mathbb{C}_+$ parametrize such invariant subspaces.

7. Let U be an inner function on $\mathbb{C}_+$ and consider the model space $K_U := H^2(\mathbb{C}_+) \ominus UH^2(\mathbb{C}_+)$. Show that evaluation at points of $\mathbb{C}_+$ is bounded on K_U and write down the reproducing kernel. Relate this kernel to the Cauchy kernel and to Clark measures (as far as covered in the notes).

8. (Conformal invariance and N_+) Prove that the Smirnov class N_+ is conformally invariant between $\mathbb{D}$ and $\mathbb{C}_+$. Use this to transport the generalized Phragmén–Lindelöf principle to $\mathbb{C}_+$ and work out an explicit example of a growth condition on f in $\mathbb{C}_+$ implying $f \in H^\infty(\mathbb{C}_+)$.

Reading and further exercise lists. For additional problem banks and perspectives, see Duren [13] (Chapters 1–4), Hoffman [10], and Nikolskii [2].

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