DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

Course: MA746: Fourier Analysis | **MidSem**
Instructor: Rajesh Srivastava | **Date:** September 19, 2025
Duration: 2.0 hours | **Maximum Marks:** 30

Note: Answers lacking rigorous justification will not be awarded marks.

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1. (a) Let $f \in L^\infty(S^1)$ with f' monotone. Does the Fourier series of f converge uniformly? **[1]**

   (b) Suppose $f \in L^1(S^1)$ and $f = g^2$. Does this imply that $\hat{g}(n) \to 0$ as $|n| \to \infty$? **[1]**

   (c) Does there exist a non-zero Lebesgue-integrable periodic function f on S^1 such that $f * (f + f') = f''$? **[1]**

2. Let $f(x, y)$ be periodic in y, with $f \in C([-\pi, \pi] \times [-\pi, \pi])$. For each fixed x, define

$$A_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y) e^{-iny}\, dy.$$

   Show that A_n converges uniformly to 0. **[3]**

3. For $f \in L^\infty(S^1)$, prove that $\lim\limits_{n\to\infty} \frac{1}{2\pi} \int\limits_{S^1} f(t)\, f(nt)\, dt = \big(\hat{f}(0)\big)^2$. **[4]**

4. Suppose $f : [-\pi, \pi] \to \mathbb{R}$ is defined by $f(t) = 1 - |t|$ if $|t| < 1$ and $(t) = 0$ if $|t| \geq 1$. Show that the kth partial sum of the Fourier series of f is

$$S_k(f)(t) = \frac{1}{2\pi} + 2 \sum_{n=1}^{k} \frac{1 - \cos(n)}{n^2 \pi} \cos(nt).$$

   Determine whether $S_k(f'')$ converges uniformly. **[4]**

5. Let $f \in C^1(S^1)$. Show that there exists $M > 0$ such that $|\hat{f}(n)| \leq \frac{M}{1+|n|}$. Furthermore, prove that $\sum\limits_{n \in \mathbb{Z}} (1 + |n|^2)\, |\hat{f}(n)|^2 < \infty$. **[2]**

6. For $f, g \in L^2(\mathbb{R})$, show that $\widehat{f * g} = \hat{f}\, \hat{g}$. **[4]**

7. Let $x^j f \in L^1(\mathbb{R})$ for $0 \leq j \leq k$. Prove independently (without using known result in this regard) that $\hat{f} \in C^k(\mathbb{R})$, and that $D^k \hat{f}(x) = \widehat{(-ix)^k f}(x)$, where $D = \frac{d}{dx}$. **[5]**

8. Let X be the subspace of $C^\infty(\mathbb{R})$ consisting of functions f such that $x^k f(x)$ is bounded for every non-negative integer k. Show that $\widehat{X} = X$. **[5]**

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