MA 101 (Mathematics-I)

Multivariable Calculus part 2: Practice Problem Sheet 1

1. Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be defined by $f(x, y) = \varphi(x)\psi(y)$, where $\varphi : [a, b] \to \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$ are continuous. Show that
$$\iint_D f(x, y)dxdy = \left(\int_a^b \varphi(x)dx\right)\left(\int_c^d \psi(x)dx\right).$$

2. Let $f : D = [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by
$$f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}^c \cap [0, 1]; \\ 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \text{ and } y \in \mathbb{Q}^c \cap [0, 1]; \\ 1 - \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest term and } y \in \mathbb{Q} \cap [0, 1]. \end{cases}$$
Then f is integrable and $\iint_D f(x, y)dxdy = 1$. Does repeated integral $\int_0^1 \left(\int_0^1 f(x, y)dy\right)dx$ exist?

3. Find the volume of the tetrahedron T bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $x - y - z = -1$.

4. Evaluate the following iterated integrals applying Fubini's Theorem.
   (a) $\int_0^1 \int_{x=y}^1 \cos(x^2)dxdy$.
   (b) $\int_0^1 \int_{y=\sqrt{x}}^1 e^{y^3}dydx$.
   (c) $\int_0^1 \int_{y=x^2}^1 x^3e^{y^3}dydx$.
   (d) $\int_0^1 \int_{x=y}^1 \frac{1}{1+x^4}dxdy$.
   (e) $\int_0^1 (\tan^{-1}\pi x - \tan^{-1} x)dx$.

5. Let D be the region lying below the curve $y = \cos x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and above the X-axis. Evaluate $\iint_D \sin x dxdy$.

6. Let D be the region in $\mathbb{R}^2$ bounded by the curves $y = 2x^2$ and $y = 1 + x^2$. Evaluate the double integral $\iint_D (2x^2 + y)dxdy$.

7. Evaluate $\iint_D x\cos\left(y - \frac{y^3}{3}\right)dxdy$, where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.

8. Find the volume of the solid enclosed by the surfaces $z = 6 - x^2 - y^2,\ z = 2x^2 + y^2 - 1,\ x = -1,\ x = 1,\ y = -1$ and $y = 1$.

9. Let D be the solid bounded by the surfaces $y = x^2,\ y = 3x,\ z = 0$ and $z = x^2 + y^2$. Find the volume of the solid.

10. Let D be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z = 1$ and $z = 0$. Find the volume of the solid.